EXPERIMENTAL TESTS OF MATHEMATICAL ABILITY AND THEIR PROGNOSTIC VALUE

BY

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TEACHERS COLLEGE, COLUMBIA UNIVERSITY
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TESTS OF MATHEMATICAL ABILITY AND THEIR PROGNOSTIC VALUE

CHAPTER I

SUMMARY OF PREVIOUS WORK

THE decision of every question of moment in education depends upon both psychological and sociological considerations. As regards the latter, investigation is greatly complicated at the present time by the far-reaching character of the changes that are taking place in our industrial and social life. So rapid and so complex are these changes that recommendations based upon yesterday's situation may prove ill-adapted to that of to-day. In this respect the psychologist has a considerable advantage over the sociologist, where educational guidance is concerned; for, however variable and elusive it may be, the original nature of man is a more stable thing than the environment to which it is exposed. The task of discovering what can be known of the innate abilities of the individual presents fewer obstacles to the scientific investigator and once attained it will remain a permanent possession and unfailing fingerpost for the educator.

In no sphere is this knowledge more desirable and necessary at the present time than in the high school subjects and particularly in mathematics. Reforms of a far-reaching character are already planned or in process and it is important that such psychological considerations as bear upon a satisfactory scheme should be ascertained. In our present comparative ignorance of the abilities involved in mathematical work, we lack one important means of estimating the significance of the reconstructions proposed in this field.

This study is a partial contribution towards supplying this need. Its purpose is to make an analysis of the abilities involved

in high school mathematics, to determine their efficiency and status, their interrelations and also their connection with certain other forms of mental capacity. Primarily it is directed to discover dynamic and quantitative relations between mathematical abilities rather than to show *how* we think in mathematics from the standpoint of analytic or structural psychology. It is to be distinguished, therefore, from the work on the thought processes of the Würzburg School, since it does not attempt to analyze the content of thought in mathematical thinking, while it seeks to determine the functional relations between mathematical abilities and their connection or lack of connection with certain other mental abilities. It is not concerned with the development of ideas of number or space in the child, which recent writers on the Psychology of Mathematics have considered at some length [1] and which has importance for the elementary school teacher. Neither is it an investigation of mathematical genius, nor even a consideration of the capacities called into play in higher mathematics, though there are certainly important features common to high school mathematics and higher mathematics. If we exclude arithmetic, which has of late received considerable attention, we find with few exceptions that most of the publications upon the psychology of the subject have treated of creative ability in mathematics and of the nature of the capacities demanded by higher mathematics and further that the balance of opinion favors the view that there is a radical difference between high school mathematics and higher mathematics. For example, Betz [2] asserts that "School mathematics has extremely little to do with real mathematical thinking." (*Nun hat aber die Schulmathematik mit dem eigentlich mathematischen Denken nur aüsserst wenig zu tun.*) Henri Poincaré [3] in like vein writes, "Many children are incapable of becoming mathematicians, to whom, however, it is necessary to teach mathematics. All we can do is to work with them, adapting ourselves to their properties." By mathematician Poincaré does not mean those possessing creative genius alone; under the term he includes those capable of

[1] See Bibliography, Howell, H. B., The Pedagogy of Arithmetic, New York, 1914.

[2] Betz, W., Uber Korrelationen, *Ztsch. für Ang. Psych.*, beihefte: 1911.

[3] Poincaré, H., The Foundations of Science, New York, 1913, tr. by Halsted, G. B.

understanding higher mathematics, though they cannot do original work in that field. William Brown similarly affirms:[4] "There is good reason for thinking that school mathematics and higher mathematics relate to different forms of ability and should be clearly distinguished from one another." These writers further contend, as likewise does Katz,[5] in reviewing the whole subject, that whereas higher mathematics demands special ability, any intelligent child can master the mathematics required in the secondary school, provided he exerts himself earnestly.

The various treatises upon mathematical genius that have appeared have utilized the methods of observation and introspection. They present two main strands of thought. On the one hand, ability in mathematics is held to be a special fundamental capacity, independent of other mental capacities—a view taken by Möbius,[6] for example. On the other hand, it is regarded merely as consisting in an "unusual ease in performing certain thought operations."

A variety of opinions exists as to the nature of these fundamental processes involved in mathematical thinking. According to Wundt[7] the essence of geometrical ability is the union of concrete imagination with deductive understanding. This particular combination produces the analyzing type of mind, characteristic of scientists and geometricians. The ability to synthesize together with inductive ability makes the discoverer, while the former coupled with deductive ability makes the speculative thinker. Mathematicians may belong to either class.

In 1894 Professor Calkins[8] communicated to the *Educational Review* a study of the Mathematical Consciousness by Wellesley students. The main conclusions reached were as follows: "Concrete memory characterizes the mathematically inclined and belongs to geometricians to a greater extent than to algebraists. Though imagination is the foundation of every mathematical as of every conscious process and though memory is at least as common among mathematicians as among average individuals, the

[4] Brown, W., The Psychology of Mathematics, *Child Study*, 6: 26.
[5] Katz, D., Psychologie und mathematischer Unterricht, Leipzig, 1913.
[6] Möbius, P. J., Uber die Anlage zur Mathematik, Leipzig, 1900.
[7] Wundt, W., Grundzüge der Physiologische Psychologie, III: 636.
[8] Calkins, M. W., A Study of the Mathematical Consciousness, *Educational Review*, VIII: 269.

essential characteristic of the student of mathematics is the power of thought, of identification, comparison and reasoning. The ability to notice the similarity or dissimilarity between objects or relations and to classify them accordingly is prominent. In algebra the given problem must be classified as one to whose solution certain rules apply. In geometry a theorem must be demonstrated. This classifying power is strong in mathematics. Further, geometry is more congenial to the true mathematician than algebra, and mathematics involves the possession of every sort of ability."

In 1900 Möbius [9] suggested that mathematical talent is characterized by exceptional ability in understanding relations of number, in judging relations of size, and in concrete imagery.

In 1905 there appeared in the French mathematical journal *l'Enseignement Mathématique* the first of a series of articles published at intervals until 1908 under the general title, "Enquête sur la Méthode de Travail des Mathématiciens." These consisted of replies to a questionnaire summarized by H. Fehr and others. They contained many interesting facts about the mental habits and methods of work of mathematicians, but suffer from the psychological superficiality of such studies.

In 1908 Henri Poincaré published a suggestive discussion of *L'Invention Mathématique* in the *Bulletin de l'Institut général Psychologique*. This was the forerunner of several papers upon the nature of mathematical ability. In these Poincaré [10] contends that "it is impossible to study the works of the great mathematicians or even those of the lesser, without noticing and distinguishing two opposite tendencies, or rather two entirely different kinds of minds. The one sort are above all preoccupied with Logic. . . . The other sort are guided by intuition. . . . The method is not imposed by the matter treated. Though one often says of the first that they are *analysts* and calls the other *geometers*, that does not prevent the one sort remaining analysts even when they work at geometry, while the others are still geometers, even when they occupy themselves with pure analysis. It is the very nature of their mind, which makes them logicians or intuitionalists, and they cannot lay it aside when they approach

[9] *Op. cit.*
[10] *Op. cit.*

a new subject. . . . Nor is it education which has developed in them one of the two tendencies and stifled the other. The mathematician is born, not made, and it seems he is born to be a geometer or analyst. . . . Among our students we notice the same differences: some prefer to test their problems by analysis, others by geometry. The first are incapable of 'seeing in space,' the others are quickly tired of long calculations and become perplexed."

Poincaré also maintains that mathematical ability is not due merely to a very sure memory nor to a prodigious power of attention. If it were, every mathematician would also be a good chess player and likewise a good computer and this is far from being the case. "In a word my memory is not bad," he writes, "but insufficient to make me a good chess player. Why does it not fail me then in a difficult piece of mathematical reasoning, where most chess players would lose themselves? Evidently because it is guided by the general march of the reasoning. A mathematical demonstration is not a simple juxtaposition of syllogisms, it is syllogisms placed in a certain order, and the order in which these elements are placed is much more important than the elements themselves. If I have the feeling, the intuition, so to speak, of this order, so as to perceive it at a glance, the reasoning as a whole, I need no longer fear lest I forget one of the elements, for each of them will take its allotted place in the array and that without any effort of memory on my part." According to Poincaré, it is this intuition of mathematical order which distinguishes the mathematician from other men.

Hüther,[11] writing in 1910, asserts that it is extraordinary development of concrete imagery, synthetic imagination and mathematical understanding that marks the mathematician, while in 1911 Betz presents the theory that the mathematical type of mind is characterized by a special clearness of certain minimal or highly abstract ideas and by the capacity to vary these with precision and to manipulate them with facility. "As soon as it gets to be a matter of discovering mathematical principles independently," he says, "then stands the unmathematical before insuper-

[11] Hüther, A., Uber das Problem einer psychologischen und pädagogischen Theorie der intellektuellen Begabung, *Archiv. für die gesamte Psychologie,* XVIII: 193.

able barriers, then he is simply incapable, and even a person of average ability as regards mathematics comes sooner or later upon a problem, which he cannot grasp without external aid and which a better mathematician can solve with relatively little effort. The mental state has a certain resemblance to the situation where one tries to hold fast in a visual image certain details, of which straightway not a significant trace is visible; but in the case of mathematical thinking it is not a matter of visual memory images, but of peculiar ideas, which are *felt* rather than *seen* and which I, in another connection, have called Minimal Ideas."

Akin to the foregoing studies is Judd's [12] treatment of the psychology of mathematics, inasmuch as it presents a survey based in part upon experimental work of the psychological processes underlying mathematics. It describes typical mental reactions involved in mathematical thinking, and analyzes the psychological implications of the text-books in use and of current class-room procedure. Judd asserts that the abilities demanded by algebra, geometry, and arithmetic are essentially different in character, each representing some forms of mental activity not included in the others.

An entirely dissimilar method of approach has been made by those workers who have been interested in the establishment of standards and scales, as likewise by those who have resorted to the use of the objective statistical method of correlations.

If we consider arithmetic, it appears that the work in this field has been extensive and has significance for mathematics in general. The most important results of the studies by Thorndike,[13] Stone,[14] Bonser,[15] Courtis,[16] Winch,[17] Starch,[18] and Woody [19] are the demonstration of the wide range of individual differences in capacity and the specialization and independence of the different abilities involved in arithmetic. A high degree of excellence in the fundamental processes (addition, subtraction, multiplication, and division) has been shown to be consistent with a low degree

[12] Judd, C. A., The Psychology of High School Subjects, New York, 1915.

[13] See Bibliography in H. B. Howell's A Foundational Study in the Pedagogy of Arithmetic, New York, 1914.

[14] *Ibid.* [15] *Ibid.* [16] *Ibid.* [17] *Ibid.* [18] *Ibid.*

[19] Woody, C., Measurements of Some Achievements in Arithmetic, Teachers College, Columbia University Contributions to Education, No. 80.

of skill in arithmetical reasoning and vice versa. Indeed a similar variability was found to prevail among the fundamental processes themselves. These results led Fox and Thorndike [20] to prophesy that the abilities tested—addition, multiplication, fractions, rational computation and problems—bear little resemblance to those of the mathematician.

Bonser [21] found similar results in investigating the reasoning ability of children in the fourth, fifth, and sixth school grades. Among others he gave certain tests of mathematical judgment. These were problems in arithmetic, stated in unusual form. He obtained the following correlations:

Arithmetic Tests and *Completion Tests*	.41
Arithmetic Tests and *Opposites*	.42
Arithmetic Tests and *Selecting Correct Reasons*	.33
Arithmetic Tests and *Selecting Best Definitions*	.26
Arithmetic Tests and *Literary Interpretation*	.26
Arithmetic Tests and *Spelling*	.24

In the field of algebra and geometry, if we exclude the efforts to establish standards for algebra by Thorndike, Monroe,[22] Rugg [23] and Clark, and Childs,[24] and standards for geometry by Stockard [25] and Carleton Bell,[26] we find that in all the experimental investigations published, with four exceptions, the data have been school and college marks or class lists. Correlations between school marks in mathematics and in English and between the former and drawing were calculated by Smith.[27] He found in

[20] Fox, W. S., and Thorndike, E. L., The Relationship between the Different Abilities involved in the Study of Arithmetic, Columbia Contributions to Philosophy, Psychology, and Education, XI: 32.

[21] Bonser, F. G., The Reasoning Ability of Children of the Fourth, Fifth and Sixth School Grades, Teachers College, Columbia University Contributions to Education, No. 37.

[22] Monroe, W. S., Measurement of Certain Algebraic Abilities, *School and Society,* I: 393, and A Test of the Attainment of First Year High School Students in Algebra, *School Review,* XXIII: 159.

[23] Rugg, H. O., The Experimental Determination of Standards in First Year Algebra, *School Review,* XXIV: 37.

Rugg, H. O. and Clark, J. E., Standardized Tests and Their Improvement of First Year Algebra, *School Review,* XXV: 115 and 196.

[24] Childs, H. G., The Measurement of Achievement in Algebra, Bulletin, Extension Division, Indiana University, II: 6.

[25] Stockard, L. V. and Bell, J. Carleton, A Preliminary Study of the Measurement of Abilities in Geometry, *Jour. Ed. Psych.* VII: 567.

[26] *Ibid.*

[27] Columbia University Contributions to Philosophy, Psychology, and Education, XI, No. 2.

the case of mathematics and English the coefficient was .36 for boys and .43 for girls: for mathematics and drawing it was .16 for boys and .12 for girls. Burris [28] obtained between English and mathematics a correlation coefficient of .39 and between algebra and geometry a coefficient of .45.

Brinckerhoff, Morris, and Thorndike [29] used the regents' examination marks in order to avoid the influence of the pupils' looks, manners and attitude upon the teacher's judgment. All pupils considered were from the same school and had practically had the same training. The coefficients between mathematics and other secondary school subjects were positive and low. They were as follows (Burris's results are given in parentheses):

Mathematics and *English*	.09	(.39)
Mathematics and *Science*	.07	(.41)
Mathematics and *History*	.26	(.33)
Mathematics and *German*	.48	
Mathematics and *Drawing*	.02	
Mathematics and *Latin*	.31	(.40)

Rietz and Shade [30] found higher correlations with science and similar correlations with foreign languages. Between mathematics and science the coefficient obtained was .440 with a P. E. of .015 and between mathematics and languages the coefficient was .476 with a P. E. of .015.

Similar results were obtained by H. O. Rugg [31] in a more recent study.

Subjects Correlated	*Value of r*
Mathematics and Descriptive Geometry	.70
Mathematics and Modern Languages	.50
Mathematics and English	.40
Mathematics and Shop-Practice	.44
Mathematics and Shop-Practice	.38
Mechanical Drawing and Shop-Practice	.44

A statistical study carried out in the pedagogical department of

[28] *Ibid.*
[29] *Ibid.*
[30] Rietz, H. L., and Shade, J., Correlation of Efficiency in Mathematics and Efficiency in other Subjects, *The University of Illinois Studies,* VI: 301.
[31] Rugg, H. O., The Experimental Determination of Mental Discipline in School Studies, Baltimore, 1916, p. 93.

Dartmouth College under the direction of F. C. Lewis,[32] in 1905, deserves mention on account of the departure made in method. Instead of using school marks as data, tests were given in originals in geometry and in practical reasoning and the scores made in these were correlated. It may be doubted whether the tests were adequate measures of the abilities in question and the method of correlation was misleading. The pupils of each of twenty-four groups were arranged in two series, the first according to their ranking in mathematical reasoning, and the second according to their ranking in practical reasoning. It was found that of the first five mathematical reasoners from each group 63 per cent., that is, 76 persons, were at the foot of the practical reasoning series, conspicuous for their inefficiency in practical reasoning; and of the pupils at the foot of the mathematical reasoning series, 47 per cent. were conspicuous for their positions at the head of the practical reasoning series.

These results have been subjected to criticism by Rietz,[33] who points out that Lewis's conclusion that they furnish convincing evidence "that students able in mathematical reasoning are not even generally able in practical reasoning and law" is far from justified, since not only were the data relatively few, but the coefficients of correlation derived from them (.38 to .675) are both positive and significant.

None of the preceding studies made any correction for the attenuation of the coefficients of correlation due to chance inaccuracies in the original measures. The true relationships between the mathematical abilities and the other abilities investigated are probably much higher than these crude coefficients indicate. We can judge to what extent the latter would be raised by correction, from the few corrected coefficients calculated by Bonser from one of the groups he examined. In the case of the data from the boys in Grade 6A two methods of correction were applied,[34] and the following coefficients were obtained. The crude coefficients are also given for purposes of comparison.

[32] Lewis, F. C., A Study in Formal Discipline, *School Review,* XIII: 281.

[33] Rietz, H. L., On the Correlation of the Marks of Students in Mathematics and in Law, *Jour. Ed. Psych.*, VII: 87.

[34] Thorndike, E. L., Theory of Mental and Social Measurements, New York, 1913, 177.

	Gross Coefs.	*Corrected Coefs. Meth. 1*	*Corrected Coefs. Meth. 2*	*Average of Cor. Coefs.*
Arithmetic Tests and *Completion Test*	.31	.55	.37	.46
Arithmetic Tests and *Opposites*	.43	1.04	.57	.81
Arithmetic Tests and *Selecting Correct Reasons*	.00	.39	.20	.10
Arithmetic Tests and *Selecting Best Definitions*	.31	.99	.45	.72
Arithmetic Tests and *Literary Interpretation*	.25	.46	.36	.41
Arithmetic Tests and *Spelling*	.19	.50	.19	.34

The following coefficients of correlation obtained by Spearman [35] in an investigation into the nature of general intelligence are still larger. In the case of the school subjects examination marks were used as data.

	Crude	*P.E.*	*Corrected*
Mathematics and *Pitch Discrimination*	.39	.03	
Mathematics and *Pitch Discrimination* (musicians only)	.45	.03	.61
Mathematics and *Mathematics* (reliability coefs.)	.88	.01	
Mathematics and *Classics*	.70	.01	.81
Mathematics and *French*	.67	.01	.78
Mathematics and *English*	.64	.01	.74
Mathematics and *General Intelligence*			.86

An attempt to secure a more complete and detailed analysis of mathematical intelligence was made in 1910 by William Brown.[36] He used the same statistical method of correlation, obtaining his data from a school examination in algebra, geometry, and arithmetic. He corrected the papers, however, in two ways, according to ordinary school standards, and also according to a differential system of marking based upon an introspective analysis of the intellectual processes involved in answering. The latter method is perhaps even more open to criticism than the former, since there are obvious defects in the "psychologizing" of examination papers.

The following results were obtained.

[35] Spearman, C., "General Intelligence," Objectively Determined and Measured, *Amer. Jour. Psych.* XV: 275.

[36] Brown, William, An Objective Study of Mathematical Intelligence, *Biometrika*, VII: 367.

	r	P.E.
Arithmetic and Algebra	.79	.03
Geometry and Algebra	.66	.04
Geometry and Arithmetic	.58	.05
Memory of preceding propositions and power of applying them and Recognition of necessity of generality of proof and power to recognize general relations in a particular case	.81	.02
Accuracy in Arithmetic and Accuracy in Algebra	.69	.04
Memory of preceding propositions and power of applying them and Power to do sums in percentage and proportion	.59	.05
General memory of rules and power to apply in Arithmetic and General memory of rules and power to apply in Algebra	.49	.06
Power to do sums in percentage and proportion and General memory of rules and power to apply in Algebra	.49	.06
Recognition of necessity of generality of proof and power to recognize general relations in a particular case and Power to do sums in percentage and proportion	.44	.06
Memory of constructions in Geometry and power to do sums in percentage and proportion	.26	.07

Brown's principal conclusions were that geometrical and algebraic ability are not related, save through their connection with arithmetical ability, that memory of preceding propositions is the central ability in geometry, being related most intimately to other forms of geometrical ability, that the difference between geometrical ability and algebraic ability justifies Poincaré's theory that mathematical reasoners fall into two distinct types, the geometrical or intuitional and the analytical or logical, and that the "balance of evidence seems to be in favor of the existence of a special capacity or faculty underlying mathematical ability, distinct from and with no essentially close connections with other forms of intellectual capacity." [37]

In 1913 appeared a study of great practical interest, T. L. Kelley's [38] "Educational Guidance." Using the method of the

[37] Brown, W., An Objective Study of Mathematical Intelligence, *Biometrika*, VII: 352 and The Psychology of Mathematics, *Child Study*, VI: 26.

[38] Kelley, T. L., Teachers College, Columbia University Contributions to Education, No. 71.

Regression Equation, the author showed how prognosis of ability in mathematics, English, and history could be made on a basis of past school record, teachers' estimates of ability, and the results of tests in these subjects. Of these three means of prognosis past school record was found to be most satisfactory, inasmuch as the prognoses so derived corresponded most closely with the actual future achievements of the individuals tested. It may, however, be objected, even when allowances are made for the difficulty of measuring abilities in a field which is new to the persons examined, that the particular mathematical tests used in this study were far from adequate measures of geometrical and algebraic abilities and that with better tests the relative values of school marks and tests as means of prognosis might be reversed. In general, the possible independence of abilities, which superficially seem closely akin, and the possible identity of abilities which superficially seem notably disparate, demonstrate the need for a many-sided gauge of mathematical ability. Existing statistical studies unequivocally suggest that here we should act on the principles of dynamic psychology, upon which Alfred Binet [39] relied in measuring general intelligence. We cannot assume that a single test or even two or three tests can give an adequate measure. The only safe and sure method is to cover as many phases of mathematical skill and insight as possible and pool the results. Theorists relying upon introspection have sought after a single clue, a distinguishing mark, which would differentiate mathematical ability from all other forms of ability and constitute the essence of mathematical talent, but this would seem to be a questionable assumption and one which itself demands experimental investigation.

This brief summary of the literature serves to show the present position of our knowledge in this field. It will be seen that it offers little more than a number of suggestions as to the nature of mathematical capacity and but scanty evidence as to the dynamic connections between mathematical abilities and other abilities. The results obtained by those using the methods of introspection and observation are for the most part speculative

[39] Binet, A., Les Idees Modernes sur les Enfants, Paris, 1911, 117 and 242, and *L'Année Psychol.* XVII, 183.

in character. Möbius, Poincaré, Hüther, Betz and others have advanced certain theories as to what constitutes the essence of mathematical capacity, but these are only interesting hypotheses, which await confirmation. The objective method of correlation has yielded more fruitful results, but since conclusions cannot be more accurate and reliable than the data from which they are derived, the fact that school and college marks were used in the bulk of statistical studies greatly limits their value and significance. Where more accurate investigation has been attempted, notably by William Brown, the ideals of scientific measurement have not been fully attained.[40]

The experiments forming the subject matter of this study are directed towards furnishing an answer to some of the more immediate and pressing problems in this field. Thus the nature of mathematical ability demands further investigation. It is important both from the theoretical and practical standpoint that the dynamic relations between mathematical abilities should be more accurately known. Without this information, control over the learning process is greatly limited. We need to determine whether there is one outstanding ability, which is the fundamental capacity in mathematical work. We require to know whether mathematical talent is such that it can function with approximately equal facility in relation to all kinds of material or whether it is in its very nature specialized, and tied down to definite objects and situations. We also stand in need of some means of making prognoses of mathematical efficiency or insufficiency with a view to educational guidance.

Thus the scope of this study is comprehended by a consideration of the following questions:

1. The development of tests which are reliable measures of the principal forms of mathematical activity required in high schools.
2. The kind and amount of correlation of these forms of activity with one another.
3. The relative value of each test as a measure of mathematical ability.

[40] See Thorndike, E. L., Theory of Mental and Social Measurements, New York, 1913, Chap. II.

4. The determination of the characteristics in a test which make for high correlation with mathematical ability.
5. The selection of a group of tests which will give a sufficiently accurate prognosis of mathematical ability.

CHAPTER II

GENERAL CONDITIONS OF THE PRESENT INVESTIGATION, APPLICATION OF THE TESTS AND SYSTEM OF SCORING*

THE SUBJECTS

THE subjects who were examined in the present investigation comprised:

(1) A group of fifty-three girls attending the Wadleigh High School. Their ages ranged from twelve and a half years to sixteen years and eight months, the average age being fourteen and a half years. They had had five months' training in formal algebra, but no geometry. The first application of the tests was made in the third week in June, 1916. As a rule the tests were given in the regular mathematics hour or in a study period, save in the case of certain tests of language ability, which were administered during the English class time. In general, the groups tested numbered twenty-five to thirty and the duration of examination was thirty-five minutes. All the mathematics tests were given by the writer, as also were the tests of verbal ability with the exception of the Thorndike Reading Scale Alpha 2 and the Trabue Language Completion Scales L and M, which have been so carefully standardized in method of application that the difference in results due to different experimenters is negligible.[1]

The second application of the tests was made in October, 1916. The summer vacation had intervened and had been considerably extended owing to the epidemic of infantile paralysis in New York City. Little further training in mathematics, therefore,

* In the making of the tests helpful constructive suggestions and criticism were given by Miss Livia. Ferrin.

[1] The writer's thanks are due to Dr. Lorle Stecher, who administered the tests of verbal ability and to Miss Helen D. Romer, who rendered helpful assistance in the distribution and collection of the tests.

had been received. In consequence of the late reopening it was necessary to make the second application of the tests after school hours. A small sum of money was offered to the subjects to induce them to stay voluntarily. Two and a half hours' testing on two afternoons from 2:30 to 5 P. M. completed their examination. On these occasions precautions were taken to avoid fatigue by conducting short breathing exercises at intervals of forty minutes, the usual duration of a class period. The interest of the group was remarkable. The girls entered into the work with zeal and earnestness. The difference in the conditions of the two applications will have to be remembered, however, when we come to consider the results and compare them with those obtained from the second group tested. In all probability they effected a reduction in the coefficients of correlation derived.

(2) A group of sixty-one pupils in the Horace Mann High School for Girls. They ranged in age from twelve years and ten months to sixteen years and eleven months, the average age being fourteen and a half years. Two-thirds of the group had had five months of intuitional geometry and five months of algebra. As in the case of the Wadleigh High School group, each test was given in duplicate, the second application generally following twenty-four hours after the first and never more than a week later. The tests were given either in the regular mathematics hour, when approximately twenty-five girls were examined together, or in a study period, in which the group as a whole participated. For this, as for the former group, all the tests were administered by the writer with the exception of the Thorndike Reading Tests, which were given by the English teacher as an English class exercise. In the case of both groups the scores obtained were made known to the pupils and considerable interest was shown in these.

Tests With Their Administration and Scoring

The tests used in this study were selected or devised to touch as many forms of mathematical achievement as possible in the particular groups examined. They can be divided into three chief classes. Six are tests of algebraic abilities and with these may be grouped a test of skill in problems in arithmetic and a test of ability to reason with symbols. The latter involves the selec-

tion of relevant data in order to deduce the required conclusion and is thus akin to the type of reasoning which predominates in algebra. The second class consists of five tests of geometrical abilities, three of which measure intuitive grasp of spatial relations, one the ability to infer with spatial data and one the power to generalize from spatial facts.

Several of the tests, it will be seen, resemble ordinary class-room exercises, save that they are arranged in an order of increasing difficulty and were applied under controlled conditions. The other mathematical tests were designed to measure abilities which obviously play a part in higher mathematics or which previous psychological investigators have stated to be essential factors in mathematical ability.

In the case of each new test, prolonged preliminary trials [2] were made and as a result some of the tests were discarded as unreliable or impracticable. Those were retained which gave the clearest indication of being adequate measures of abilities important in the mental equipment of the student of mathematics. Eventually the following tests were adopted.

1. Algebraic Computation
2. Matching Equations and Problems
3. Matching Nth Terms and Series
4. Interpolation
5. Missing Steps in Series
6. Inference with Symbols
7. Geometry
8. Superposition
9. Symmetry
10. Matching Solids and Surfaces
11. Geometrical Definitions
12. Arithmetic Problems
13. Reasoning

In addition to these tests of mathematical activities a third series of tests of language ability was given. The purpose underlying their application was to discover how far weakness in mathematics depends upon or is connected with inferiority in command of the vernacular. The tests of language ability used were the following:

1. Mixed Relations
2. Logical Opposites
3. Trabue Language Scales
4. Thorndike Reading Tests

The coefficients of reliability obtained from correlating the

[2] The writer is indebted to Principal J. Cayce Morrison for the opportunity to make these preliminary trials of the tests in Chatham High School.

two applications of the same test in the case of the Wadleigh High School group were in several instances too low to be satisfactory. This was accounted for in part by the differences in the conditions of application, but was also attributed to remediable defects in the tests, such as their short duration. They were therefore extended not only in time, but also in difficulty, before being applied to the second group.

A brief description of the tests applied follows.

Algebraic Computation Test:

The following test was constructed for the purpose of measuring efficiency in algebraic computation. After each problem are given directions for grading it.

ALGEBRAIC COMPUTATION TEST

(1)

1. Let C stand for the cost and SP for the selling price and G for the gain. Then $G=SP-C$.

   What is G if C is $10 and SP is $12.50?
   Ans.................. (Score 1 or 0)

2. Let L stand for the length of a room and W stand for the width and SF for the area. Then $SF=L\times W$.
   What is SF when $L=18$ feet and $W=10$ feet?
   Ans.................. (Score 1 or 0)

3. If $a=2$ and $b=3$ and $c=5$ and $d=1$, write the values of:

   $5a$ *Ans*.................. (Score 1 or 0)

   $2a-d$ *Ans*.................. (Score 1 or 0)

   $\dfrac{a+b+d}{d}$ *Ans*.................. (Score 1 or 0)

   $\dfrac{2a+c}{3d}$ *Ans*.................. (Score 1 or 0)

   $\dfrac{2c}{a}-\dfrac{4b}{3d}$ *Ans*.................. (Score 1 or 0)

4. $2¢+5¢-3¢+9¢-2¢=$ how many ¢?
   Ans.................. (Score 1 or 0)

5. $3a+4a+7a-5a+6a=$ how many a's?
   Ans.................. (Score 1 or 0)

6. $3x+3y+4z+2z+2x=$ how many x's, how many y's and how many z's?
   Ans.................. (Score 3, 2, 1, or 0)

7. $3x+2y-3z+7z-2x+9y+3x=$how many x's, how many y's and how many z's?
Ans..................(Score 3, 2, 1, or 0)

8. If $6x=30$, what does $x=$? *Ans*..................(Score 1 or 0)

9. If $x-2=5$, what does $x=$? *Ans*..................(Score 1 or 0)

10. If $2x+3=15$, what does $x=$? *Ans*..................(Score 1 or 0)

11. If $15-x=9$, what does $x=$? *Ans*..................(Score 1 or 0)

12. If $x=4$ and $2x=5+y$, what does $y=$? *Ans*...........(Score 1 or 0)

13. A man is now 40 years old, how old was he 8 years ago?
Ans..................(Score 1 or 0)

14. Y stands for the number of years in a man's age now, how old was he 5 years ago? *Ans*..................(Score 1 or 0)

15. If L stands for the length of a room, what is the length of a room 4 feet longer? *Ans*..................(Score 1 or 0)

16. If 1 pencil costs 5 cents, how many cents will B pencils cost?
Ans..................(Score 1 or 0)

ALGEBRAIC COMPUTATION TEST

(1a)

Multiply and remove parenthesis:

1. $4(3x-4)=$
Ans...............
(Score 2, 1, or 0)

2. $-5(-4x-6y)=$
Ans................
(Score 2, 1, or 0)

3. $-4(x-2)=$
Ans...............
(Score 2, 1, or 0)

Change all terms containing x to the left side of the equation and all others to the right side.

1. $-17x-12=192-7x+32$
Ans..........................
(Score 2, 1, or 0)

2. $17x-38+7=33x-8x-91$
Ans...........................
(Score 2, 1, or 0)

Clear the following equations of fractions. Do *not* collect or transpose terms.

1. $$\frac{-7x-2}{6}=\frac{x+1}{8}-x$$
Ans......................
(Score 2, 1, or 0)

2. $$\frac{x-3}{9}-\frac{5x+4-1}{12}=0$$
Ans..........................
(Score 3, 2, 1, or 0)

Solve the following problems for x.

1. $$\frac{-3x-2}{4} = \frac{x+2}{6}$$

2. $$4x - \frac{2(-4x+7)}{8} = 3 + \frac{3(3x-2)}{5}$$

Ans............(Score 3, 2, 1, or 0) *Ans*............(Score 2, 1, or 0)

Find the values of both unknowns in the following equations:

1. $7x-4y=12$
   $8x-5y=0$

Ans.: $x=$ .... $y=$ ....(Score 3, 2, 1, or 0)

2. $4x-3y=1$
   $3x-4y=6$

Ans.: $x=$ .... $y=$ ....(Score 3, 2, 1, or 0)

3. $5x+9y=28$
   $7x+3y=29$

Ans.: $x=$ .... $y=$ ....(Score 3, 2, 1, or 0)

Matching Equations and Problems Test:

This test was designed to measure the ability to translate verbal statements of problems into algebraic symbolism. In form it is a matching test and has thus the advantage of isolating the task of translation from other factors. It has the added merit of presenting a familiar process in a novel form. The facility with which the subject can cope with the new situation affords some indication of the degree of mathematical intelligence he possesses rather than a measure of efficiency in a habitual method of working. The score equals the number of problems correctly matched.

Page 1

Name............................ Date............................

An Equation is a short method of writing a problem. Here is a Problem and an Equation which stands for it.

Problem: If 7 is subtracted from a certain number, the remainder is 13; what is the number?

Equation $x-7=13$

On the other side of this sheet there are 10 Problems and 10 Equations which stand for them. Pick from the 10 Problems the one which is represented by the first Equation and write its number in COLUMN 1 opposite the Equation. Do the same for the other Problems and Equations.

Page 2

Instructions: In column 1 write opposite each Equation the number of the problem which it stands for. Do not write any number twice. Write only one number opposite each letter. Omit no number. Do not find the answers to the problems. Only pair the Equations and Problems.

PROBLEMS:

1. I had \$20 in my purse when I went down town and \$6 when I returned. How much did I spend?
2. I earned \$6 to-day and now have \$20. How much did I have this morning?
3. If six times a certain number is 20, what is the number?
4. Each member of a class of 20 buys a copy of a book. If the class spends \$6, how much is the book per copy?
5. Find a number such that one-sixth of it equals 20.
6. John and Mark both have marbles, but John has six more than twice as many as Mark. If John has 20 marbles, how many has Mark?
7. Six less than twice a certain number is 20, what is the number?
8. Find a number such that if the number is subtracted from 20, the result obtained is the same as if 6 had been added to the number.
9. I have twice as much money as John has. If I spend \$20 and he spends \$6 we will have the same amount. How much money has John?
10. Find a number such that if 20 is subtracted from twice the number the result is 6.

Column 1	*Equations*
________	A. $20-x=6+x$
________	B. $x+6=20$
________	C. $2x+6=20$
________	D. $6x=20$
________	E. $2x-20=x-6$
________	F. $\frac{x}{6}=20$
________	G. $2x-6=20$
________	H. $20-x=6$
________	I. $20x=6$
________	J. $2x-20=6$

Matching Nth Terms and Series Test:

The material for this test consists in a group of arithmetical

progressions and of corresponding formulae. These are arranged in haphazard order and the subject has to match them correctly. The nature of the test was carefully explained, much time being spent in making certain that it was understood. Each formula correctly matched was awarded 1 mark. A series of 12 formulae and corresponding arithmetical progressions was given. This was followed immediately by a second series of 20 formulae, and 20 progressions in the case of the Horace Mann group.

Page 1

Name............................ Date............................

DIRECTIONS

If in the Formula $2n$ we let n=first 1, then 2, then 3, then 4, then 5, then 6, then 7, we shall get first 2, then 4, then 6, then 8, then 10, then 12, then 14; that is the Series of numbers 2, 4, 6, 8, 10, 12, 14. . . . . . Similarly if in the Formula $5n—1$ we again let n=1, 2, 3, 4, 5, 6, 7 in turn, we shall get the Series 4, 9, 14, 19, 24, 29, 34. . . . . .

Therefore

Series	*Formula*
2, 4, 6, 8, 10, 12, 14	is obtained from $2n$
4, 9, 14, 19, 24, 29, 34	is obtained from $5n—1$

On the other side of this sheet there are twelve such Series and twelve Formulae from which they were obtained by letting n=first 1, then 2, then 3, then 4, then 5, then 6, then 7.

The Series and Formulae have to be paired. Pick from the 12 Series the one which is obtained from the first Formula and write in the empty column, called *Column* 3 its number opposite the Formula. Do the same for the other Series and Formulae.

Page 2

REMEMBER: Write in *Column* 3 opposite each Formula the number of the Series that is obtained from it.
Write only one number opposite each Formula.
Do not write any number twice.

	Series								*Formulae*	*Column 3*
(1)	3	7	11	15	19	23	27	....	$n+5$	
(2)	1	7	13	19	25	31	37	....	$3n$	
(3)	6	11	16	21	26	31	36	....	$4n—1$	
(4)	6	12	18	24	30	36	42	....	$5n+1$	
(5)	6	7	8	9	10	11	12	....	$6n$	
(6)	1	3	5	7	9	11	13	....	$n—1$	
(7)	3	6	9	12	15	18	21	....	$n+2$	

(8)	0	1	2	3	4	5	6	$2n-1$	
(9)	6	10	14	18	22	26	30	$3n-3$	
(10)	0	3	6	9	12	15	18	$7n-5$	
(11)	3	4	5	6	7	8	9	$6n-5$	
(12)	2	9	16	23	30	37	44	$4n+2$	

				Series				*Formulae*	*Column 3*
(1)	8	16	24	32	40	48	....	$n+7$	
(2)	8	14	20	26	32	38	....	$10n-2$	
(3)	8	17	26	35	44	53	....	$2n+2$	
(4)	4	12	20	28	36	44	....	$2n+6$	
(5)	4	7	10	13	16	19	....	$12n-4$	
(6)	8	9	10	11	12	13	....	$5n-1$	
(7)	8	19	30	41	52	63	....	$7n-3$	
(8)	8	23	38	53	68	83	....	$8n$	
(9)	8	11	14	17	20	23	....	$3n+1$	
(10)	4	6	8	10	12	14	....	$8n-4$	
(11)	4	11	18	25	32	39	....	$9n-1$	
(12)	8	10	12	14	16	18	....	$14n-6$	
(13)	8	18	28	38	48	58	....	$3n+5$	
(14)	8	15	22	29	36	43	....	$15n-7$	
(15)	8	20	32	44	56	68	....	$4n+4$	
(16)	8	13	18	23	28	33	....	$11n-3$	
(17)	8	22	36	50	64	78	....	$7n+1$	
(18)	8	12	16	20	24	28	....	$13n-5$	
(19)	4	9	14	19	24	29	....	$6n+2$	
(20)	8	21	34	47	60	73	....	$5n+3$	

Interpolation Test:

The material for this test consists in arithmetical series from which certain steps have been omitted and which have to be replaced.

The test was introduced in the hope that it would give some indication of the pupil's ability to analyze numerical or symbolic data, to perceive a general rule implicit in them and to apply the principle so derived. One mark was given for each blank correctly filled. The blanks increase in number as the series grows in difficulty. The test was extended in length for the Horace Mann group.

Page 1

Name.............................. Age..............................

DIRECTIONS: Do not turn this page until the signal is given!!
Stop working at once when you hear *stop*!!

The following is a series of numbers, in which each number follows the one before it according to a rule.

2 4 6 8 10 12 14 16

Thus each number is got from the one before it by adding 2 to it. If any of the numbers in the series is missing, it is possible to replace it.

For example, in the series: 2 4 — 8 10 — 14
the missing numbers are: 6 and 12

Similarly, in the series: 5 10 — 20 25 — 35
the missing numbers are: 15 and 30

Similarly, in the series: 1 4 — 10 — — — 22
the missing numbers are: 7, 13, 16, and 19

On the other side of this page there are series similar to those above, from which certain numbers are missing. You must supply the missing numbers. Your score depends on the number of blanks correctly filled.

Page 2

(1)

(A)	1	3	5	7	—	11	13	15	17	—	21		
(B)	1	5	9	13	—	21	25	29	33	—	41		
(C)	0	3	6	9	—	15	18	21	24	—	30		
(D)	1	8	15	—	29	36	43	—	57	64	71		
(E)	7	13	19	25	—	37	43	—	55	61	67		
(F)	5	13	—	29	37	45	53	61	—	77	85		
(G)	3	12	21	—	39	48	57	—	75	84	93		
(H)	2	—	8	—	14	—	20	—	26	—	32		
(J)	0	—	8	—	16	—	24	—	32	—	40		
(K)	7	—	15	—	23	—	31	—	39	—	47		
(L)	3	—	—	18	—	—	33	—	—	—	53		
(M)	4	—	—	13	—	—	22	—	—	—	34		
(N)	5	—	—	—	33	—	—	—	61	—	—		
(P)	8	—	—	—	28	—	—	—	48	—	—		
(Q)	—	—	13	—	—	—	25	—	—	—	37		
(R)	—	—	13	—	—	—	29	—	—	—	45		
(S)	2	—	—	—	—	37	—	—	—	—	72		
(T)	11	—	—	—	—	66	—	—	—	—	121		
(U)	7	—	—	—	—	—	43	—	—	—	—	—	79
(V)	7	—	—	—	—	—	31	—	—	—	—	—	55

(1a)

(A)	1	8	15	22	—	36	43	50	—	64	71
(B)	3	7	11	15	—	23	27	31	—	39	43
(C)	4	10	16	22	—	34	40	46	—	58	64
(D)	6	10	14	—	22	26	30	—	38	42	46
(E)	9	20	31	—	53	64	75	—	97	108	119
(F)	5	17	29	—	53	65	77	—	101	113	125

(G)	8	17	26	—	44	53	—	71	80	89	98		
(H)	0	—	16	—	32	—	48	—	64	—	80		
(J)	7	—	13	—	19	—	25	—	31	—	37		
(K)	2	—	16	—	30	—	44	—	58	—	72		
(L)	1	—	—	7	—	—	13	—	—	—	21		
(M)	4	—	—	31	—	—	58	—	—	—	94		
(N)	7	—	—	—	27	—	—	—	47	—	—		
(P)	9	—	—	—	57	—	—	—	105	—	—		
(Q)	—	—	13	—	—	—	29	—	—	—	45		
(R)	—	—	16	—	—	—	44	—	—	—	72		
(S)	3	—	—	—	—	33	—	—	—	—	63		
(T)	2	—	—	—	—	47	—	—	—	—	92		
(U)	6	—	—	—	—	—	72	—	—	—	—	—	138
(V)	5	—	—	—	—	—	83	—	—	—	—	—	161

Missing Steps in Series Test:

This test is similar to the previous one. In this case, however, the examples given involve the four common arithmetical processes of addition, subtraction, multiplication, and division. A preliminary test containing one example of each type, was first given. The score depended, as in the Interpolation Test, upon the number of blanks correctly filled.

Page 1

Name............................ Date............................

DIRECTIONS

Each of the lists of numbers below is a *Series,* in which each number follows the one before it according to a rule. Thus in the

Series 5 10 15 20 25 30 ....
each number is obtained from the one before it by adding 5 to it.
(5+5=10; 10+5=15; 15+5=20; and so on.) Similarly in

Series 78 72 66 60 54 48 ....
each number is obtained from the one before it by subtracting 6 from it.
(78—6=72; 72—6=66; 66—6=60; and so on.) Similarly in

Series 2 6 18 54 162 486 ....
each number is obtained from the one before it by multiplying it by 3.
(2×3=6; 6×3=18; 18×3=54; and so on.) Similarly in

Series 128 64 32 16 8 4 ....
each number is obtained from the one before it by dividing it by 2.
(128÷2=64; 64÷2=32; 32÷2=16; and so on.)

The Series of numbers on the other side of this sheet are obtained in

similar ways. You are to find the rule for each Series and so supply the missing numbers, as in the examples below.

Examples	(a)	5	10	*15*	20	25	30
	(b)	78	72	*66*	60	54	48
	(c)	2	6	*18*	54	162	486
	(d)	128	64	*32*	16	8	4

Page 2

Series	(1)	1	3	—	7	9	11
	(2)	16	13	—	7	4	1
	(3)	2	4	—	16	32	64
	(4)	1	8	—	22	29	36
	(5)	32	16	—	4	2	1
	(6)	1	4	—	64	256	1024
	(7)	6250	1250	—	50	10	2
	(8)	44	36	—	20	12	4
	(9)	7	13	—	25	31	37
	(10)	7	14	—	56	112	224
	(11)	26	21	—	11	6	1
	(12)	1701	567	—	63	21	7

Inference with Symbols Test:

The object of this test was to ascertain how far the ability to manipulate symbols with ease and precision correlates with mathematical ability. For this purpose five common symbols were chosen, of which only one, the sign for "equals," was familiar to the subjects. The task was to make inferences with regard to the relation of a pair of terms, when information about their relations with mediating terms had been given. Several illustrative examples were shown before the test was applied. Two series were administered, forming a scale of increasing difficulty. The score equalled the number of correct inferences made.

This test was considerably extended in the case of the Horace Mann group both as regards time and difficulty.

Page 1

Name .... Date............................

Number.............

DIRECTIONS

Using the facts under *GIVEN FACTS*, fill in the blank spaces under *FILL IN* with $>$, $<$, $=$, $\not>$ or $\not<$, whichever gives the true conclusion. In any case where none of the symbols can be correctly used, make a —— in the blank space.

$>$ means "greater than"
$<$ means "less than"
$=$ means "equal to"
$\ngtr$ means "not greater than"
$\nless$ means "not less than"

Read this illustration, which will show you what is to be done.

GIVEN FACTS		*FILL IN*
$a > b = c$	therefore	$a > c$
$a > b \nless c$	therefore	$a > c$

Page 2

(1)

REMEMBER
$>$ means "greater than"
$<$ means "less than"
$=$ means "equal to"
$\ngtr$ means "not greater than"
$\nless$ means "not less than"
— means "where none of the other symbols fit"

	GIVEN FACTS		*FILL IN*
1.	$a = b = c$	therefore	a c
2.	$a = b > c$	therefore	a c
3.	$a < b = c$	therefore	a c
4.	$a = b \nless c$	therefore	a c
5.	$a \ngtr b = c$	therefore	a c
6.	$a < b < c$	therefore	a c
7.	$a \ngtr b \ngtr c$	therefore	a c
8.	$a < b > c$	therefore	a c
9.	$a > b \ngtr c$	therefore	a c
10.	$a < b \ngtr c$	therefore	a c
11.	$a \nless b < c$	therefore	a c
12.	$a \ngtr b < c$	therefore	a c

(1a)

(1)	$a > b = c = d$	therefore	a d
(2)	$a = b = c > d$	therefore	a d
(3)	$a = b < c = d$	therefore	a d
(4)	$a > b > c = d$	therefore	a d
(5)	$a = b < c < d$	therefore	a d
(6)	$a \ngtr b = c = d$	therefore	a d
(7)	$a = b = c \ngtr d$	therefore	a d
(8)	$a = b \nless c = d$	therefore	a d
(9)	$a \ngtr b \ngtr c = d$	therefore	a d

(10) $a > b > c > d$ therefore $a \qquad d$
(11) $a = b \nless c \nless d$ therefore $a \qquad d$
(12) $a \nless b \nless c \nless d$ therefore $a \qquad d$
(13) $a < b < c > d$ therefore $a \qquad d$
(14) $a < b \ngtr c < d$ therefore $a \qquad d$
(15) $a > b < c < d$ therefore $a \qquad d$
(16) $a < b > c > d$ therefore $a \qquad d$
(17) $a > b > c \nless d$ therefore $a \qquad d$
(18) $a > b \nless c > d$ therefore $a \qquad d$
(19) $a = b \ngtr c < d$ therefore $a \qquad d$
(20) $a > b = c \nless d$ therefore $a \qquad d$
(21) $a \nless b = c > d$ therefore $a \qquad d$
(22) $a > b > c \ngtr d$ therefore $a \qquad d$
(23) $a \nless b < c < \mathrm{d}$ therefore $a \qquad d$
(24) $a < b \nless c \nless d$ therefore $a \qquad d$
(25) $a > b \nless c \ngtr d$ therefore $a \qquad d$
(26) $a < b = c \ngtr d$ therefore $a \qquad d$
(27) $a < b \nless c < d$ therefore $a \qquad d$
(28) $a \ngtr b \ngtr c > d$ therefore $a \qquad d$
(29) $a < b \ngtr c > d$ therefore $a \qquad d$
(30) $a \nless b \nless c \ngtr d$ therefore $a \qquad d$

Geometry Test:

The material for this test consists in a series of geometrical principles together with a number of geometrical problems, whose solution depends upon the former. The task is to solve the problems with the aid of the principles. In order to acquaint the pupils with the requirements of the test a preliminary problem was first given and its solution demonstrated: this brought to light any cases of misunderstanding of the directions. The method of grading the test is indicated after each problem.

Page 1

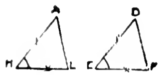
THIS SHEET IS FOR REFERENCE ONLY

FACTS GIVEN AS TRUE

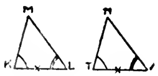
(1) *All right angles have 90 degrees.*
This is a right angle.

(2) *All straight angles have 180 degrees.*
This is a straight angle.

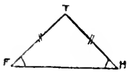
(3) *All the angles of a triangle added together equal 180 degrees.*
Thus in triangle AKL, angle A, angle K and angle L added together equal 180 degrees.



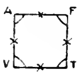
(4) *Two triangles are equal if two sides and the angle between them are equal.*
Thus the triangle AHL equals triangle DEF since side AH equals side DE and side HL equals side EF, and angle H equals angle E.



(5) *Two triangles are equal if two angles and the side between them are equal.*
Thus triangle KLM equals triangle TVN, since angle K equals angle T and angle L equals angle V and side LK equals side TV.

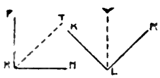


(6) *An isosceles triangle has two sides equal and the angles opposite them equal.*
Thus FTH is an isosceles triangle, since FT equals TH and angle F equals angle H.



(7) *The sides of a square are equal, and all its angles are right angles.*
Thus $AVTF$ is a square, since AV equals VT equals TF equals FA, and angles A, V, T and F are right angles.

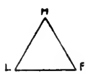
(8) *If two lines are equal, their halves are equal.*
Thus AC equals DF, therefore BC (half of AC) equals EF (half of DF).



(9) *If two angles are equal their halves are equal.*
Thus angle FHN equals angle KLM, therefore THN (half of angle FHN) equals angle VLM (half of angle KLM).

Directions: Taking the facts (1), (2), up to (9) as true, do the work required on the other sheet and write your answer in the space reserved. Do not take anything for granted not given in (1), (2), etc. above or under *Given* on the other sheet. First do 1, then do 2, and so on. In every case when you use any of the facts above (1), (2), etc. in your work, write the number to show what fact it is you use.

Example



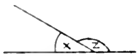
Given: Angle L=60 degrees.
The triangle is isosceles.
Prove: Angle F= ? degrees.

Answer:

The triangle is isosceles (Given).
Therefore angle L equals angle F by fact (6).
Angle L equals 60 degrees (Given).
Therefore angle F equals 60 degrees.

Page 2

1.

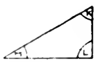


Given: Angle $X=30$ degrees.
Prove: Angle $Z=?$ degrees.

Answer:

Angles X and $Z=180$ degrees (Score 1) by fact 2 (Score 1).
Therefore Angle $Z=$ 150 degrees (Score 1).

2



Given: Angle L is a right angle.
Prove: Angle $K=?$ degrees.
Angle $M=30$ degrees.

Answer:

Angle $L=90$ degrees (Score 1) by fact 1 (Score 1).
Angles K, L, and $M=180$ degrees (Score 1) by fact 3 (Score 1).
Therefore $K=60$ degrees (Score 1).

3.

Given: The triangle is isosceles.
Angle $P=30$ degrees.
Prove: Angle $Q=?$ degrees.

Answer:

Angles P and Q are equal (Score 1) by fact 6 (Score 1).
Therefore Angle $Q=30$ degrees (Score 1).

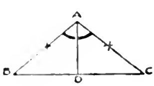
4

Given: The triangle is isosceles.
Angle $A=30$ degrees.
Prove: Angle $X=?$ degrees.

Answer:

Angles A and D are equal (Score 1) by fact 6 (Score 1).
Angles X and D together$=180$ degrees (Score 1) by fact 2 (Score 1).
Therefore Angle $X=150$ degrees (Score 1).

5.

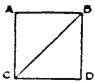


Given: The triangle is isosceles.
The line from A to D is drawn so as to make angle $BAD=$angle DAC.
Prove: The two small triangles equal.

Answer:

Angle B=Angle C (Score 1) by fact 6 (Score 1).
Angle BAD=Angle DAC, Given (Score 1).
Line BA=Line AC (Score 1) by fact 6 (Score 1).
Therefore the 2 triangles are equal by fact 5 (Score 1).

6.



Given: The figure is a square.
Two opposite corners are joined.
Prove: The square is divided into equal triangles by the line drawn.

Answer:

Line AB=Line BD (Score 1) by fact 7 (Score 1).
Line AC=Line CD (Score 1) by fact 7 (Score 1).
Angle A=Angle D (Score 1) by fact 7 (Score 1).
Therefore the 2 triangles are equal by fact 4 (Score 1).
(Corresponding system of marking for proof by fact 5.)

Superposition Test:

This test was developed by L. L. Thurstone, of the Carnegie Institute of Technology, Pittsburgh, as a measure of the ability to grasp spatial relations. In the present investigation it served two purposes. It measured the dexterity with which the subject could apply both the principle of superposition and that of symmetry. It consists essentially of pairs of symmetrical parallelograms, each with one side on the same straight, long, black line and each adjoining a third parallelogram of corresponding design, and similarly with one black edge, but such that it can be superposed upon only one of the adjoining parallelograms. This third parallelogram which has a small circle in one corner is placed in a variety of positions relative to the pair of parallelograms. The task, in this case, was to imagine the third parallelogram moved around so that it fitted one of the corresponding pair of parallelograms and to indicate which it was by drawing a small circle in the corner of the same, exactly where the circle in the third parallelogram would then lie.

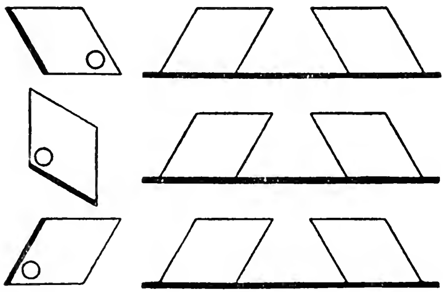
The score equals the number of circles correctly placed. This

test was applied twice to the Wadleigh High School group and four times to the Horace Mann group.

Symmetry Test:

The material for this was the same as the foregoing, while the method was changed. The subject had to imagine the third parallelogram picked up, turned over and placed face down with its black edge touching the long heavy, black line to the right. The card was then imagined to be moved until its edges fitted the edges of one or other of the two parallelograms. A circle had to be drawn in the corner where the circle in the third parallelogram would then lie. The score equals the number of circles correctly marked. This test was extended when applied to the Horace Mann group.

The following examples are typical:



Matching Solids and Surfaces:

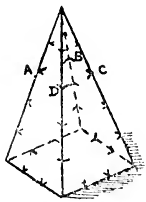
The purpose of this test was similar to that of the two preceding with this difference that it was expressly directed towards estimating the facility with which tri-dimensional relations were intuitively grasped. In form it resembles a matching test, the subject having to name all the solids from which the given surfaces could be obtained by a single cross-section and to indicate the nature of the transection necessary. For each solid correctly named a score of 1 was given; for each section of correct shape indicated an additional mark was awarded, and where a

section of correct size as well as shape was indicated two additional marks were assigned.

REFERENCE SHEET:

These are drawings of Solids.

(1)



(2)

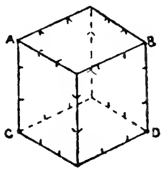
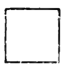


Fig. 1 is 2 inches high and its base is 1 square inch in size.
Fig. 2 is 1 cubic inch in size.

These are drawings of Surfaces.

(3)



(4)

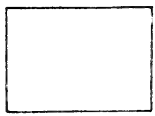


Fig. 3 is half an inch square. It is obtained by cutting Solid 1 *straight* through *once* through the points *A, B, C and D.*

Fig. 4 is one inch high and one and two-fifths inches broad and is obtained by cutting Solid 2 *straight* through *once* through the points *A, B, C* and *D.*

On the other sheet there are seven drawings of Solids and below these there are seven drawings of Surfaces obtained by cutting these Solids *straight* through *once* with a flat knife.

Solid I. Altitude 2 inches; Base 1 inch square.
Solid II. Altitude 2 inches; Base, diameter 1 inch.
Solid III. Altitude 2 inches; Base 1 inch on each side.
Solid IV. Diameter 1 inch.
Solid V. Altitude 2 inches; Base 1 inch square.
Solid VI. Altitude 2 inches; Base, diameter 1 inch.
Solid VII. Major axis one and one-half inches; Minor axis 1 inch.

DIRECTIONS

Examine Surface 1. Decide on *all* the Solids from which it can be obtained by cutting them straight through once with a flat knife in *any* direction. Write in the empty space below the Surface the numbers of the Solids from which it can be obtained. Also show how the Solids must be cut by lettering the points on the *edges* of the Solids through which the *knife must pass.* Write these letters after the corresponding number of the Solid in the space below. Do the same for Surfaces 2, 3, 4, 5, 6 and 7.

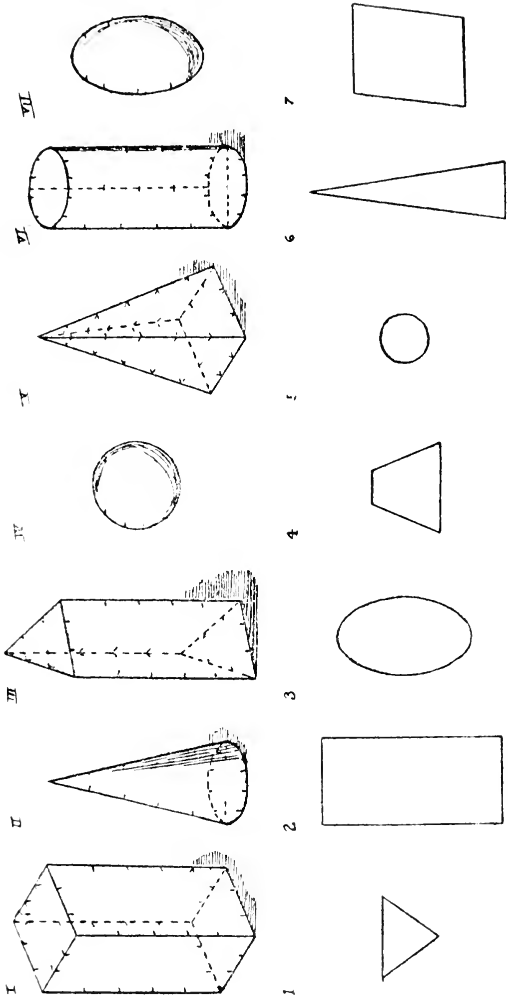
REMEMBER: (1) Give the points through which the knife must pass; do not give the points which outline the required Surface.
(2) The points through which the knife passes must give the Surface correct in both shape and size.
(3) The same Surface may be obtained from several Solids.
(4) No Surface should be obtained more than once from the same Solid.
(5) One point has only one letter-name.
(6) The effect of perspective.

[*See cut opposite page*]

Geometrical Definitions Test:

This a modified form of a test by Winch.[3] After reading carefully a series of illustrative examples, the subjects had to write definitions for geometrical figures of various kinds, which were shown. The ability to analyze common and also differentiating features and to generalize from spatial data was measured by this test. The method of marking followed was that used by its originator.

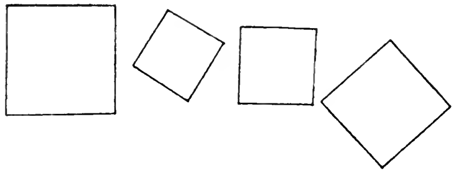
[3] See Winch, W. H., Inductive *versus* Deductive Methods of Teaching: An Experimental Research, Baltimore, 1913.



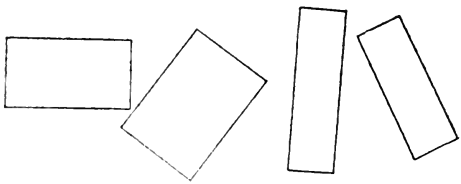
Matching Solids and Surfaces. *(See page 34 for description of this test.)*

GEOMETRICAL DEFINITIONS TEST

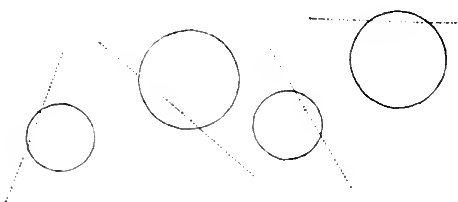
Preliminary Examples



1. These are Squares, therefore a Square is a figure with 4 equal straight lines as sides and 4 equal angles which are right angles.

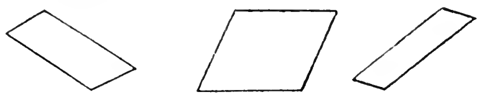


2. These are Rectangles, therefore a Rectangle is a figure bounded by 4 straight lines, of which the opposite sides are equal and parallel and whose angles are all equal and right angles.

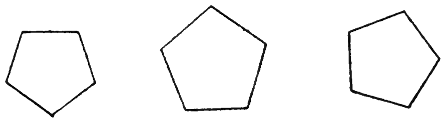


3. These dotted lines are Secants, therefore a Secant is a straight line that cuts the circumference of a circle in 2 points.

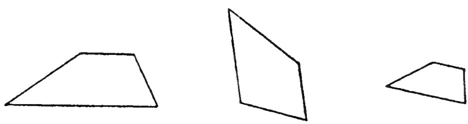
Geometrical Definitions:



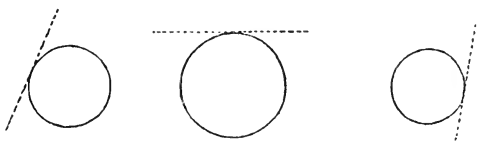
1. These are Rhomboids; give a complete definition of a Rhomboid.



2. These are regular Pentagons; give a complete definition of a regular Pentagon.



3. These are Trapezoids; give a complete definition of a Trapezoid.



4. These dotted lines are Tangents; give a complete definition of a Tangent to a circle.

Reasoning Test:

This test was devised on lines suggested by Thorndike. It consists of a ladder-like arrangement of inferences, of gradually growing complexity and difficulty. It was designed to measure the ability to seize the relevant elements in a complex and abstract situation and to respond only to them. The score depended upon the number of correct inferences drawn.

Page 1

Name.............................. Date..............................

DIRECTIONS

On the other side of this sheet fill in the blank spaces under *fill in* with conclusions which can be correctly drawn from the facts stated under *given facts.*

Read this illustration which will show what is to be done.

Given Facts		*Fill In*
R is shorter than *S*	therefore	*T* is *longer than R.*
T is longer than *S*		
R is longer than *V*	therefore	*V* is *shorter than S.*

Page 2

	Given Facts		*Fill In*
(1)	*P* is longer than *Q*		
	R is shorter than *Q*	therefore	*P* is................*R*
(2)	*M* is younger than *N*		
	K is older than *N*	therefore	*K* is................*L*
	M is older than *L*	therefore	*N* is................*L*
(3)	*M* is richer than *O*		
	O is as rich as *P*	therefore	*M* is................*K*
	K is poorer than *P*	therefore	*N* is................*M*
	N is poorer than *K*		
(4)	*Z* is thicker than *X*	therefore	*X* is................*H*
	H is as thick as *Z*	therefore	*Y* is................*H*
	V is thicker than *H*	therefore	*X* is................*V*
	V is thinner than *Y*	therefore	*Z* is................*Y*
(5)	*D* is greater than *B*	therefore	*B* is................*A*
	B is equal to *E*	therefore	*D* is................*F*
	E is greater than *F*	therefore	*E* is................*A*
	C is less than *F*	therefore	*B* is................*C*
	A is greater than *D*	therefore	*A* is................*F*

(6) *A* is higher than *G*	therefore	*A* is...............*D*
B is equal to *E*		
D is lower than *G*	therefore	*F* is...............*B*
C is higher than *G*		
G is lower than *B*	therefore	*H* is...............*E*
B is hegher than *H*		
A is lower than *H*	therefore	*D* is...............*C*
F is equal to *H*		
E is higher than *G*	therefore	*E* is...............*F*

Graded Problems in Arithmetic:

This test was constructed by Thorndike. It is self-explanatory. One mark was given for each answer, if absolutely correct.

Name............................ Date.............................

GRADED PROBLEMS (1)

Find how long Mary was allowed to play on each of these days.

Answers

1. Monday. It is 4.10 P. M. Supper is at 6 o'clock. Mother says "you may play half the time from now till supper time." __________

2. Wednesday. It is 4.05 P. M. Mother says "if you help me for half an hour now, and for 10 minutes before supper you may play the rest of the afternoon." __________

3. Friday. Mother says you may play 2 minutes for every three problems you solve, and five minutes more for every problem you solve correctly. Mary solved 15 and has all but one right. __________

The rest of these problems all ask the same question: how many minutes is it from the time John begins to pump until the tank is filled? The tank holds 120 gallons and is supposed always to be empty when John begins to work.

Answers

4. John pumps 2 minutes before any water reaches the tank. Then he pumps water into it at the rate of 3 gallons a minute until the tank is full. __________

5. John pumps 8 minutes before any water reaches the tank. Then he pumps water into it at the rate of 24 gallons in 10 minutes until the tank is full. __________

6. John pumps 1¾ minutes before any water reaches the tank. Then he pumps for 10 minutes at the rate of 2.7 gallons a minute. Then the pump breaks and he spends 8 minutes mending it. Then he pumps at the rate of 3.1 gallons per minute until the tank is full. ______

Thorndike Reading Tests: [4]

The Thorndike Reading Scale Alpha 2 was used for the first measure of the ability to understand sentences and a number of passages of corresponding complexity for the second. The standard method of marking was adopted [4] in the case of Scale Alpha 2. For the passages, the scale of marking was 4, 3, 2, 1, 0.

Tests of Verbal Ability

1. *Mixed Relations Test:* [5]

This is a well-known test, in which the task is to discover a fourth term, which stands in the same relation to the given third term as the second does to the first. Twenty such examples were presented to the Wadleigh High School girls and double that number to the Horace Mann group. Three marks were given for a perfect score and two or one for partially correct solutions according to the degree of the correctness.[6]

2. *Logical Opposites Test:* [7]

In this test the subject was given a list of thirty words in the case of the Wadleigh High School students and one hundred words in the case of the Horace Mann group. The logical opposite had in each instance to be written. The score for a perfect answer was 3, while 2 or 1 was given for efforts of a less appropriate kind according to the degree of correctness.

[4] See *Teachers College Record*, September, 1914, November, 1915, and January, 1916.

[5] See Whipple, G. M., Manual of Mental and Physical Tests, 2d ed., 1914, Pt. 2: 89-94, for a description of the use that has been made of the Mixed Relations Test.

[*See page* 41 *for footnote* 6]

[7] For the previous application of this test the reader is referred to the same source, 79-89.

3. *Trabue Language Scales:*[8]

These consist of a series of sentences from which certain words have been omitted. The sentences are graded in difficulty and standardized. Scales L and M were given to the Wadleigh High School group and these together with Scales J and K to the Horace Mann pupils. The method of scoring used was that followed in standardizing the scales.

[6] The material used in the Mixed Relations Test was the following:

eye—see	ear—
Monday—Tuesday	April—
do—did	see—
bird—sings	dog—
hour—minute	minute—
straw—hat	leather—
cloud—rain	sun—
hammer—tool	dictionary—
uncle—aunt	brother—
dog—puppy	cat—
little—less	much—
wash—face	sweep—
house—room	book—
sky—blue	grass—
swim—water	fly—
once—one	twice—
cat—fur	bird—
pan—tin	table—
buy—sell	come—
oyster—shell	banana—
past—present	present—
come—came	go—
north—south	far—
mend—clothes	bake—
lily—flower	oak—
ton—pound	pound—
elbow—arm	chin—
pea—pod	nut—
bell—rings	clock—
deep—valley	high—
growls—dog	roars—
brick—wall	page—
lathe—machine	hammer—
pencil—lead	book—
high—low	up—
sheep—lamb	dog—
kettle—brass	cup—
Thursday—Friday	June—
build—house	paint—
one—single	two—
mice—cat	worms—
London—England	Paris—
A church organ—banjo	Hamlet—

A corresponding duplicate series was arranged.

[8] See Trabue, M. R., Completion-Test Language Scales, Teachers College, Columbia University Contributions to Education, No. 77.

CHAPTER III

THE ANALYSIS OF MATHEMATICAL ABILITY

FROM the foregoing description one may judge to what extent the tests are representative of the activities involved in high school mathematics and how far the essential data for the theoretical analysis of mathematical intelligence have been secured. Whatever their limitations and defects, it can hardly be doubted that these tests of algebraic and geometrical abilities give valuable information about the intellectual efficiency of pupils in first-year mathematics and further independent objective evidence will be presented later to show that they actually do measure important elements in the mental equipment of the prospective student of the subject. Together with the tests of language ability, which give auxiliary aid in interpreting results, they furnish a workable instrument for experimental analysis and research. For purposes of practical diagnosis ease in administration is essential. The tests must be convenient as well as typical and comprehensive. Their practicability may likewise be judged by the preceding description, together with the detailed instructions to be found in the Appendix.

It is further necessary, if the tests proposed are to be considered satisfactory, that they should be reliable measuring rods of the abilities investigated. We have an objective indication of the reliability of a test, when two distinct series of measurements by the same test of the same group give similar results. Thus for a test to be scientific and trustworthy, the relative positions of the individuals examined should be the same on every application. On the other hand, if chance is exercising a preponderating influence upon the results, slight correspondence between several trials will be found. The amount of such correspondence between any two applications can be given precise quantitative expression in the coefficient of correlation derived from two inde-

pendent sets of measures. This reliability coefficient is conditioned in part by the number of cases examined. If a sufficiently representative sample has been tested and the value of the reliability coefficient obtained is small, the test should obviously be reconstructed. Whenever the coefficient is less than .60 and pro-

TABLE I

Reliability Coefficient for Each Test, and for Its Two Applications Combined. Wadleigh High School

	r_1	r_2
Algebraic Computation	.77	.87
Matching Equations and Problems	.63	.77
Matching *N*th Terms and Series	.67	.80
Interpolation	.71	.83
Missing Steps in Series	.75	.86
Inference with Symbols	.23	.38
Geometry	.66	.80
Superposition	.84	.91
Symmetry	.92	.96
Matching Solids and Surfaces	.35	.52
Geometrical Definitions	.31	.47
Mixed Relations	.42	.59
Logical Opposites	.85	.79
Trabue Language Scales	.46	.63
Thorndike Reading Tests	.50	.67
Reasoning	.43	.60
Arithmetic Problems	.46	.63

Note:

r_1 is the Reliability Coefficient or coefficient of correlation between two applications of the tests.

r_2 is the Reliability Coefficient for the two applications of the tests combined. r_2 equals $\frac{2r_1}{1+r_1}$

It measures the extent to which the amalgamated results of the two applications would correlate with a similar amalgamated pair of two other applications of the same test. See Brown, William, The Essentials of Mental Measurement. Cambridge, 1911:101-102.

vided the number of cases is sufficiently large, the test ought to be improved. In many such instances it may only require extension.

The reliability coefficients for each test were first calculated for the Wadleigh High School group. They are summarized in Table I. The numerical data upon which they are based, as also all the other results of this investigation, are recorded in the Appendix in Tables XXXIV and XXXV in which the original scores are given.

Six of the coefficients are under .50 and even when allowance was made for their attenuation by the different conditions in the two applications and the length of time intervening, to which we have already referred, it seemed desirable that the tests should be lengthened and improved, wherever that was practicable, before further application was made.

With a view to determining from the ascertained reliability coefficients the number of applications of any particular test, which would be necessary to give an amalgamated result of approximately perfectly reliability, the formula suggested by William Brown [1] was used,

$$r_n = \frac{nr_1}{1 + (n - 1)r_1}$$

where n represents the number of applications of a test and r_1 is the coefficient of correlation between any two applications.

Besides being prolonged in accordance with this guiding rule, several of the tests were amended as regards method of application and increased in difficulty, so that a scale better fitted to differentiate degrees of ability was evolved. Owing to lack of time, all of the extensions prepared could not be applied. While the following tests were given in identical form to both the Wadleigh and the Horace Mann groups,—Algebraic Computation, Matching Equations and Problems, Missing Steps in Series, Geometry Test, Matching Solids and Surfaces, Geometrical Defi-

[1] Brown, William, The Essentials of Mental Measurement, Cambridge, 1911, 101-102.

nitions, Reasoning Test, Arithmetic Problems, and the Thorndike Reading Tests, the eight remaining tests were administered in the extended form in which they now appear.

The reliability coefficients obtained from the second application are presented in Table II.

TABLE II

RELIABILITY COEFFICIENT FOR EACH TEST, AND FOR ITS TWO APPLICATIONS COMBINED. HORACE MANN SCHOOL

	r_1	r_2
Algebraic Computation	.79	.88
Matching Equations and Problems	.61	.75
Matching *N*th Terms and Series	.81	.89
Interpolation	.94	.97
Missing Steps in Series	.70	.82
Inference with Symbols	.82	.90
Geometry	.75	.86
Superposition	.82	.90
Symmetry	.96	.98
Matching Solids and Surfaces	.70	.82
Geometrical Definitions	.84	.91
Mixed Relations	.79	.88
Logical Opposites	.58	.73
Trabue Language Scales	.33	.49
Thorndike Reading Tests	.57	.73
Reasoning	.73	.85
Arithmetic Problems	.81	.76

NOTE:

r_1 is the Reliability Coefficient or coefficient of correlation between two applications of the tests.

r_2 is the Reliability Coefficient for the two applications of the tests combined. r_2 equals $\frac{2r_1}{1+r_1}$

It measures the extent to which the amalgamated results of the two applications would correlate with a similar amalgamated pair of two other applications of the same test. See Brown, William, The Essentials of Mental Measurement. Cambridge, 1911:101-102.

It will be seen that in all the tests of mathematical abilities whose reliability coefficients on the former application were conspicuously low, there is a marked increase in reliability, while of the three tests of verbal ability, which were doubled or more than doubled in length, only one, Mixed Relations, shows any improvement. The causes for this can be traced to the unfavorable experimental conditions. Whereas all the tests in algebra and geometry were made in a regular class period of forty minutes duration, these three tests of language ability were applied in a short twenty-minute period from 9 to 9:20 A. M., in which the pupils are usually given an opportunity to consult with their section-teacher, should that be necessary. Consequently there was neither the same readiness nor concentration of attention that characterized their behavior in the remainder of the tests.

We know, however, that when the conditions of experimentation are favorable, these tests furnish adequate reliability coefficients and as they are primarily introduced in this study not for use in isolation, but *en masse* as a measure of language efficiency, the reliability coefficients yielded by their amalgamated results are sufficiently high to be satisfactory.

The increase in the amount of the reliability coefficients of the mathematics tests in the second as compared with the first group of pupils examined is largely due to the fact that irrevelant factors were excluded to a much greater extent. The second application of each test of the same mental function, for example, was usually given at the same hour of the following day and the time devoted to testing was invariably thirty-five to forty minutes.

Having determined the amount of confidence to which the tests are entitled, we can now consider the extent of connection which they reveal between the various functions examined. This is best shown by the quantitative expression of correspondence in coefficients of correlation. The standard "Product-Moments method," discovered by Bravais in 1846 and demonstrated by Pearson in 1896 to be the most satisfactory, has been used throughout this study. The formula in its most convenient form is as follows:

$$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \Sigma y^2}}$$

In this r is the required correlation, x and y are the deviations of any pair of characteristics from their respective central tendencies, $\Sigma\, xy$ is the sum of such products for all individuals, $\Sigma\, x^2$ is the sum of the squares of all the values of x, $\Sigma\, y^2$ is the sum of the squares of all the values of y. Where any existing positive relationship is observed between two traits, the coefficient assumes some value between 0 and +1; where an existing inverse or negative relationship is found, the value of the coefficient lies somewhere between 0 and —1. The coefficient thus may have any value from +1 through 0 to —1, according as the relationship is present in some amount or absent and according as the correspondence is positive or negative in nature.

Before we can attribute evidential value to the coefficients of correlation obtained by means of the above formula, however, it is essential to determine their probable errors due to the fact that only a limited sample of the total number of high school pupils has been examined. Some measure of the variability in results that we must expect, should other groups of individuals be tested, has to be provided. A sufficiently accurate formula for determining the probable error of a coefficient of correlation, when the number of cases is fairly large and the distribution of frequencies is normal has been suggested by Pearson.

$$\text{P.E.} = .67449 \frac{(1 - r^2)}{\sqrt{n}}$$

This defines the limits within which a coefficient may vary in value by accident. Its meaning may be seen from the statement that the chances are even that the true value of the coefficient r lies between the limits

$$r \pm \frac{.67449\,(1 - r^2)}{\sqrt{n}}$$

Each coefficient of correlation will then have to be compared with its probable error in order to ascertain whether it demonstrates any actual interdependence of the functions in question. For a coefficient to be considered satisfactory evidence of an

existing correspondence it has to be several times larger than its Probable Error. No coefficient less than twice as large can establish any conclusion about the actual existence of functional interdependence between two abilities. In Tables III and IV, the Probable Errors for the Wadleigh High School group and the Horace Mann School group have been calculated for the various values of r by the formula,[2]

$$\text{P.E.} = .6744898 \frac{(1 - r^2)^{6}}{\sqrt{n}}$$

TABLE III

PROBABLE ERROR OF THE COEFFICIENTS OF CORRELATION:

Wadleigh High School (n=53)

r	P.E.
.9	.02
.8	.03
.7	.05
.6	.06
.5	.07
.4	.08
.3	.08
.2	.09
.1	.09

TABLE IV

PROBABLE ERROR OF THE COEFFICIENTS OF CORRELATION:

Horace Mann School (n=61)

r	P.E.
.9	.02
.8	.03
.7	.04
.6	.06
.5	.06
.4	.07
.3	.08
.2	.08
.1	.09

We are now in a position to examine critically the results obtained from the application of the statistical methods described above to the two sets of data which form the basis of this study. Coefficients of correlation were calculated separately for the two groups, since spurious correlation would have arisen, had their records been mingled.[3] In Tables V and VI, the coefficients of correlation between each application of each test and a corresponding application of every other test are summarized for the Wadleigh High School and the Horace Mann School groups re-

[2] Winifred Gibson's "Tables for Facilitating the Computation of Probable Errors" in *Biometrika*, IV: 385, were used and David Heron's "Abac to determine the Probable Errors of Correlation Coefficients" in *Biometrika*, VII: 411.

[3] See Yule, G. Udny, An Introduction to the Theory of Statistics, London, 1916, 218-219.

spectively. The coefficients found between the two applications of every test and age are likewise included. In Tables VII and VIII corresponding pairs of coefficients in the two preceding tables are amalgamated so that each coefficient in Tables VII and VIII is the average of two corresponding applications of a pair of tests. Table IX combines the coefficients from the two groups, giving double weight to the Horace Mann results in virtue of their greater reliability. The tables of amalgamated coefficients present the results in a more comprehensible form and facilitate their interpretation.

These correlation tables furnish the data for an analysis of mathematical ability. The subtle interrelations of the complex capacities, which we vaguely indicate by the term mathematical intelligence, are already partially revealed in these figures and closer scrutiny will disclose more fully the nature of the correspondences that hold between the various functions involved.

TABLE V

Crude Coefficients—Wadleigh High School

		Algebraic Computation	Algebraic Computation	Matching Equations and Problems	Matching Equations and Problems	Matching Nth Terms and Series	Matching Nth Terms and Series	Interpolation	Interpolation	Missing Steps in Series	Missing Steps in Series	Inference with Symbols	Inference with Symbols	Geometry	Geometry
		1	2	1	2	1	2	1	2	1	2	1	2	1	2
Algebraic Computation	1				.38		.16		.34		.61		—.11		.26
Algebraic Computation	2			.54		.07		.37		.49		.04		.31	
Matching Equations and Problems	1		.54				.11		.02		.34		.01		.21
Matching Equations and Problems	2	.38				.23		.26		.24		.01		.22	
Matching Nth Terms and Series	1		.07		.23				.17		.01		.25		.03
Matching Nth Terms and Series	2	.16		.11				.04		.23		.01		.19	
Interpolation	1		.37		.26		.04				.37		.31		.39
Interpolation	2	.34		.02		.17				.30		.15		.07	
Missing Steps in Series	1		.49		.24		.23		.30				.19		.15
Missing Steps in Series	2	.61		.34		.01		.37				—.23		.00	
Inference with Symbols	1		.04		.01		.01		.15		—.23				.21
Inference with Symbols	2	—.11		.01		.25		.31		.19				.15	
Geometry	1		.31		.22		.19		.07		.00		.15		
Geometry	2	.26		.21		.03		.39		.15		.21			
Superposition	1		.28		.31		.29		.19		.21		.15		.06
Superposition	2	.26		.11		.10		.33		.19		.12		.05	
Symmetry	1		.10		.19		.18		.10		.02		.14		.25
Symmetry	2	—.05		.15		—.01		.08		.07		.21		.20	
Matching Solids and Surfaces	1		.36		.18		.19		.10		.21		.10		.07
Matching Solids and Surfaces	2	.16		.02		.08		.22		.23		.01		.38	
Geometrical Definitions	1		.31		.06		.01		.27		.16		.32		.33
Geometrical Definitions	2	.11		.17		.17		—.03		.10		—.02		.41	
Reasoning	1		.21		.25		—.08		.38		.24		.21		.13
Reasoning	2	.31		.19		.06		.32		.17		.14		.06	
Arithmetic Problems	1		.02		.15		—.12		.40		.20		.14		.10
Arithmetic Problems	2	.06		.26		—.05		.25		.26		.10		.24	
Mixed Relations	1		.01		.32		—.15		.06		.04		.06		.13
Mixed Relations	2	.19		.32		—.11		.26		.16		.20		.34	
Logical Opposites	1		.28		.29		.26		.15		.31		.24		.23
Logical Opposites	2	.28		.30		—.02		.32		.28		.31		.18	
Trabue Language Scales	1		.29		.20		.21		.02		.08		.17		.34
Trabue Language Scales	2	.36		.43		—.05		.21		.18		.16		.24	
Thorndike Reading Tests	1		.09		.26		.01		.19		—.17		.26		—.06
Thorndike Reading Tests	2	.13		.27		—.11		.17		—.05		.19		.39	
Age	1		.25		.17		—.00		—.39		—.15		—.24		—.04
Age	2	—.24		—.12		—.15		—.35		—.22		.02		—.23	

TABLE V—*Continued*

Superposition 1	Superposition 2	Symmetry 1	Symmetry 2	Matching Solids and Surfaces 1	Matching Solids and Surfaces 2	Geometrical Definitions 1	Geometrical Definitions 2	Reasoning 1	Reasoning 2	Arithmetic Problems 1	Arithmetic Problems 2	Mixed Relations 1	Mixed Relations 2	Logical Opposites 1	Logical Opposites 2	Trabue Language Scales 1	Trabue Language Scales 2	Thorndike Reading Tests 1	Thorndike Reading Tests 2	Age 1	Age 2
.28	.26	.10	—.05	.36	.16	.31	.11	.21	.31	.02	.06	.01	.19	.28	.28	.29	.36	—.09	.13	—.25	—.24
.31	.11	.19	.15	.18	.02	.06	.17	.25	.19	.15	.26	.32	.32	.29	.30	.20	.43	.26	.27	—.17	—.12
.29	.10	.18	—.01	.19	.08	.01	.17	—.08	.06	—.12	—.05	—.15	—.11	.26	—.02	.21	.05	.01	—.11	—.00	—.15
.19	.33	.10	.08	.10	.22	.27	—.03	.38	.32	.40	.25	.06	.26	.15	.32	.02	.21	.19	.17	—.39	—.35
.21	.19	.02	.07	.21	.23	.16	.10	.24	.17	.20	.26	.04	.16	.31	.28	.08	.18	—.17	—.05	—.15	—.22
.15	.12	.14	.21	.10	.01	.32	—.02	.21	.14	.14	.10	.06	.20	.24	.31	.17	.16	.26	.19	—.24	.02
.06	.05	.25	.20	.07	.38	.33	.41	.13	.06	.10	.24	.13	.34	.23	.18	.34	.24	—.06	.39	—.04	—.23
		.69	.52	.36	.37	.25	.14	.24	.32	.29	.09	.11	.23	.16	.01	—.01	.07	.14	—.05	—.01	—.05
.52	.69			.06	.42	.37	.38	.23	.06	.32	.23	.01	.26	—.01	.17	.07	.25	.21	.23	.06	.06
.37	.36	.42	.06			.37	.00	.31	.17	.23	—.05	.04	.31	.23	.00	.19	—.05	.19	—.05	—.26	.05
.14	.25	.38	.37	.00	.37			—.04	.06	—.16	.43	—.01	.30	.22	.31	.14	.36	—.14	.32	.06	—.15
.32	.24	.06	.23	.17	.31	.06	—.04			.16	.28	.15	.12	.22	.17	.22	.13	—.06	.02	—.05	—.21
.09	.29	.23	.32	—.05	.23	.43	—.16	.28	.16			—.09	—.01	—.01	.13	.22	.16	—.08	.12	—.25	—.13
.23	.11	.26	.01	.31	.04	.30	—.01	.12	.15	—.01	—.09			.45	.33	.21	.01	.30	—.03	—.13	.11
.01	.16	.17	—.01	.00	.23	.31	.22	.17	.22	.13	—.01	.33	.45			.23	.47	.22	.31	—.02	—.17
.07	—.01	.25	.07	—.05	.19	.36	.14	.13	.22	.16	.22	.01	.21	.47	.23			.31	.33	.03	—.02
—.05	.14	.23	21	—.05	.19	.32	—.14	.02	—.06	.12	—.08	—.03	.30	.31	.22	.33	.31			.03	.06
—.05	—.01	.06	.06	.05	—.26	—.15	.06	—.21	—.05	—.13	—.25	.11	—.13	—.17	—.02	—.02	.03	.06	.03		

TABLE VI

Crude Coefficients—Horace Mann School

		Algebraic Computation 1	Algebraic Computation 2	Matching Equations and Problems 1	Matching Equations and Problems 2	Matching Nth Terms and Series 1	Matching Nth Terms and Series 2	Interpolation 1	Interpolation 2	Missing Steps in Series 1	Missing Steps in Series 2	Inference with Symbols 1	Inference with Symbols 2	Geometry 1	Geometry 2
Algebraic Computation	1				.66		.45		.50		.58		.50		.47
Algebraic Computation	2			.59		.32		.65		.62		.43		.52	
Matching Equations and Problems	1		.59				.40		.32		.55		.28		.48
Matching Equations and Problems	2	.66				.21		.41		.52		.53		.38	
Matching Nth Terms and Series	1		.32		.21				.30		.31		.15		.18
Matching Nth Terms and Series	2	.45		.40				.38		.45		.38		.29	
Interpolation	1		.65		.41		.38				.63		.49		.53
Interpolation	2	.50		.32		.30				.73		.45		.45	
Missing Steps in Series	1		.62		.52		.45		.73				.48		.42
Missing Steps in Series	2	.58		.55		.31		.63				.40		.42	
Inference with Symbols	1		.43		.53		.38		.45		.40				.43
Inference with Symbols	2	.50		.28		.15		.49		.48				.28	
Geometry	1		.52		.38		.29		.45		.42		.28		
Geometry	2	.47		.48		.18		.53		.42		.43			
Superposition	1		.51		.34		.17		.31		.36		.28		.48
Superposition	2	.34		.34		.04		.43		.25		.32		.44	
Symmetry	1		.28		.16		.10		.24		.40		.10		.35
Symmetry	2	.21		.32		.17		.32		.23		.07		.32	
Matching Solids and Surfaces	1		.26		.09		—.04		.25		.23		.19		.38
Matching Solids and Surfaces	2	.22		.09		—.00		.27		.24		.19		.52	
Geometrical Definitions	1		.47		.33		.15		.35		.41		.32		.58
Geometrical Definitions	2	.03		.31		.01		.31		.39		.38		.57	
Reasoning	1		.36		.20		.24		.15		.21		.32		.45
Reasoning	2	.30		.30		.14		.29		.35		.40		.42	
Arithmetic Problem	1		.63		.54		.36		.45		.55		.45		.32
Arithmetic Problems	2	.41		.28		.17		.51		.48		.25		.33	
Mixed Relations	1		.36		.23		.11		.36		.48		.19		.41
Mixed Relations	2	.33		.27		.05		.32		.32		.19		.35	
Logical Opposites	1		.47		.44		.21		.31		.41		.24		.32
Logical Opposites	2	.32		.36		.14		.32		.29		.20		.28	
Trabue Language Scales	1		.12		—.04		.17		.06		.27		—.12		.22
Trabue Language Scales	2	.26		.30		.19		.25		.24		.25		.32	
Thorndike Reading Tests	1		.34		.32		.06		.26		.24		.24		.29
Thorndike Reading Tests	2	.43		.42		.29		.37		.41		.33		.45	
Age	1		.41		—.29		—.25		—.52		—.43		—.40		—.40
Age	2	.46		—.29		—.13		—.43		—.37		—.52		—.48	

TABLE VI—*Continued*

Superposition 1	Superposition 2	Symmetry 1	Symmetry 2	Matching Solids and Surfaces 1	Matching Solids and Surfaces 2	Geometrical Definitions 1	Geometrical Definitions 2	Reasoning 1	Reasoning 2	Arithmetic Problems 1	Arithmetic Problems 2	Mixed Relations 1	Mixed Relations 2	Logical Opposites 1	Logical Opposites 2	Trabue Language Scales 1	Trabue Language Scales 2	Thorndike Reading Tests 1	Thorndike Reading Tests 2	Age 1	Age 2
.51	.34	.28	.21	.26	.22	.47	.03	.36	.30	.63	.41	.36	.33	.47	.32	.12	.26	.34	.43	—.41	—.46
.34	.34	.16	.32	.09	.09	.33	.31	.20	.30	.54	.28	.23	.27	.44	.36	—.04	.30	.32	.42	—.29	—.29
17	.04	.10	.17	—.04	—.00	.15	.01	.24	.14	.36	.17	.11	.05	.21	.14	.17	.19	.06	.29	—.25	—.13
31	.43	.24	.32	.25	.27	.35	.31	.15	.29	.45	.51	.36	.32	.31	.32	.06	.25	.26	.37	—.52	—.43
36	.25	.40	.23	.23	.24	.41	.39	.21	.35	.55	.48	.48	.32	.41	.29	.27	.24	.24	.41	—.43	—.37
28	.32	.10	.07	.19	.19	.32	.38	.32	.40	.45	.25	.19	.19	.24	.20	—.12	.25	.24	.33	—.40	—.52
48	.44	.35	.32	.38	.52	.58	.57	.45	.42	.32	.33	.41	.35	.32	.28	.22	.32	.29	.45	—.40	—.48
		.63	.58	.31	.38	.41	.35	.40	.35	.36	.39	.32	.14	.14	.15	.10	.20	.01	.28	—.20	—.17
58	.63			.28	.39	.31	.24	.02	.24	.29	.31	.34	.20	.10	.63	.36	.12	.11	.31	—.09	—.14
38	.31	.39	.28			.54	.43	.45	.51	.39	.12	.30	.31	.24	.18	.22	.22	.17	.22	—.10	—.11
35	.41	.24	.31	.43	.54			.34	.35	.38	.29	.42	.39	.20	.19	.19	.42	.34	.52	—.39	—.28
35	.40	.24	.02	.51	.45	.35	.34			.40	.21	.34	.20	.38	.15	.20	.24	.37	.54	—.23	—.18
39	.36	.31	.29	.12	.39	.29	.38	.21	.40			.33	.44	.32	.24	.15	.46	.18	.47	—.28	—.26
14	.32	.20	.34	.31	.30	.39	.42	.20	.34	.44	.33			.19	.33	.28	.25	.40	.38	—.18	—.20
15	.14	.63	.10	.18	.24	.19	.20	.15	.38	.24	.32	.33	.19			.14	.28	.30	.31	—.22	—.19
20	.10	.12	.36	.22	.22	.42	.19	.24	.20	.46	.15	.25	.28	.28	.14			.36	.49	—.32	—.11
28	.01	.31	.11	.22	.17	.52	.34	.54	.37	.47	.18	.38	.40	.31	.30	.49	.36			—.45	—.27
17	—.20	—.14	—.09	—.11	—.10	—.28	—.39	—.18	—.23	—.26	—.28	—.20	—.18	—.19	—.22	—.11	—.32	—.27	—.45		

TABLE VII

Average Crude Coefficients—Wadleigh High School

	Algebraic Computation	Matching Equations and Problems	Matching Nth Terms and Series	Interpolation	Missing Steps in Series	Inference with Symbols	Geometry
Algebraic Computation		.46	[.12]	.36	.55	.	.28
Matching Equations and Problems	.46		[.17]	[.14]	.29	[.01]	.21
Matching Nth Terms and Series	[.12]	[.17]		[.10]	[.12]	[.13]	[.11]
Interpolation	.36	[.14]	[.10]		.33	.23	.23
Missing Steps in Series	.55	.29	[.12]	.33		[—.02]	[.07]
Inference with Symbols	[—.04]	[.01]	[.13]	.23	[—.02]		.18
Geometry	.28	.21	[.11]	.23	[.07]	.18	
Superposition	.27	.21	.19	.26	.20	[.14]	[.06]
Symmetry	[.03]	.18	[.10]	[.09]	[.05]	.18	.22
Matching Solids and Surfaces	.26	[.10]	[.12]	[.16]	.22	[.05]	.23
Geometrical Definitions	.21	[.11]	[.09]	[.12]	[.13]	[.15]	.37
Reasoning	.26	.22	[—.01]	.35	.21	.18	[.09]
Arithmetic Problems	.36	.21	[—.09]	.32	.23	[.12]	[.17]
Mixed Relations	[.10]	.32	[—.13]	[.16]	[.10]	[.13]	.23
Logical Opposites	.28	.29	[.12]	.24	.29	.28	.21
Trabue Language Scales	.32	.31	[.13]	[.12]	[.13]	[.16]	.29
Thorndike Reading Tests	[.02]	.26	[.09]	.18	[—.11]	.22	[.17]
Age	—.25	[—.15]	[—.08]	—.37	—.18	[—.11]	[—.14]

Coefficients of less than 2 P.E. are put in square brackets.

TABLE VIII

Average Crude Coefficients—Horace Mann School

	Algebraic Computation	Matching Equations and Problems	Matching Nth Terms and Series	Interpolation	Missing Steps in Series	Inference with Symbols	Geometry
Algebraic Computation		.63	.39	.57	.60	.46	.49
Matching Equations and Problems	.63		.30	.37	.53	.41	.43
Matching Nth Terms and Series	.39	.30		.34	.38	.26	.24
Interpolation	.57	.37	.34		.68	.47	.49
Missing Steps in Series	.60	.53	.38	.68		.44	.42
Inference with Symbols	.46	.41	.26	.47	.44		.36
Geometry	.49	.43	.24	.49	.42	.36	
Superposition	.42	.34	[.11]	.37	.31	.30	.46
Symmetry	.25	.24	[.14]	.28	.31	[.08]	.33
Matching Solids and Surfaces	.24	[.09]	[—.02]	.26	.23	.19	.45
Geometrical Definitions	.25	.32	[.08]	.33	.40	.35	.57
Reasoning	.33	.25	.19	.22	.28	.36	.43
Arithmetic Problems	.52	.41	.27	.48	.52	.34	.33
Mixed Relations	.34	.25	[.08]	.34	.40	.19	.38
Logical Opposites	.39	19	.28	.31	.35	.22	.30
Trabue Language Scales	.19	[.13]	.18	[.15]	.26	[.07]	.27
Thorndike Reading Tests	.39	37	.18	.32	.32	.28	.37
Age	.44	—.30	—.19	—.48	—.40	—.46	—.44

Coefficients of less than 2 P.E. are put in square brackets.

TABLE VII—*Continued*

Superposition	Symmetry	Matching Solids and Surfaces	Geometrical Definitions	Reasoning	Arithmetic Problems	Mixed Relations	Logical Opposites	Trabue Language Scales	Thorndike Reading Tests	Age
.27	[.03]	.26	.21	.26	[.04]	[.10]	.28	.32	[.02]	—.25
.21	.18	[.10]	[.11]	.22	.21	.32	.29	.31	.26	[—.15]
.19	[.10]	[.14]	[.09]	[—.01]	[—.09]	[—.13]	[.12]	[.13]	[.09]	[—.08]
.26	[.09]	[.16]	[.12]	.35	.32	[.16]	.24	[.12]	.18	—.37
.20	[.05]	.22	[.13]	.21	.23	[.10]	.29	[.13]	[—.11]	—.18
[.14]	.18	[.05]	[.15]	.18	[.12]	[.13]	.28	[.16]	.22	[—.11]
[.06]	.22	.23	.37	[.09]	[.17]	.23	.21	.29	[.17]	[—.14]
	.60	.36	.20	.28	.19	[.17]	[.08]	[.03]	[.05]	[—.03]
.60		.24	.37	[.14]	.27	[.14]	[.08]	[.16]	.22	[.06]
.36	.24		.19	.24	[.09]	[.17]	[.12]	[.07]	[.07]	[—.10]
.20	.37	.19		[.01]	[—.02]	[.14]	.26	.25	[.09]	[—.05]
.28	[.14]	.24	[.01]		.22	[.14]	.20	[.17]	[.02]	[—.13]
.19	.27	[.09]	[—.02]	.22		[—.05]	[.06]	.19	[.02]	—.19
[.17]	[.14]	[.17]	[.14]	[.14]	[—.05]		.39	[.11]	[.14]	[—.01]
[.08]	[.08]	[.12]	.26	.20	[.06]	.39		.35	.27	[—.10]
[.03]	[.16]	[.07]	.25	[.17]	.19	[.11]	.35		.32	[.00]
[.05]	.22	[.07]	[.09]	[.02]	[.02]	[.14]	.27	.32		[.04]
[—.03]	[.06]	[—.10]	[—.05]	[—.13]	—.19	[—.01]	[—.10]	[.00]	[.04]	

TABLE VIII—*Continued*

Superposition	Symmetry	Matching Solids and Surfaces	Geometrical Definitions	Reasoning	Arithmetic Problems	Mixed Relations	Logical Opposites	Trabue Language Scales	Thorndike Reading Tests	Age
.42	.25	.24	.25	.33	.52	.34	.39	.19	.39	—.44
.34	.24	[.09]	.32	.25	.41	.25	.19	[.13]	.37	—.30
[.11]	[.14]	[—.02]	[.08]	.19	.27	[.08]	.28	.18	.18	—.19
.37	.28	.26	.33	.22	.48	.34	.31	[.15]	.32	—.48
.31	.31	.23	.40	.28	.52	.40	.35	.26	.32	—.40
.30	[.08]	.19	.35	.36	.34	.19	.22	[.07]	.28	—.46
.46	.33	.45	.57	.43	.33	.38	.30	.27	.37	—.44
	.61	.35	.38	.37	.37	.23	[.14]	[.15]	[.14]	—.18
.61		.33	.28	.28	.29	.27	.36	.24	.21	[—.12]
.35	.33		.49	.48	.25	.31	.21	.22	.20	[—.11]
.38	.28	.49		.35	.34	.40	.19	.31	.43	—.33
.37	.28	.48	.35		.31	.27	.26	.22	.45	—.21
.37	.29	.25	.34	.31		.39	.28	.31	.32	—.27
.23	.27	.31	.40	.27	.39		.26	.26	.39	—.19
[.14]	.36	.21	.19	.26	.28	.26		.21	.31	—.21
[.15]	.24	.22	.31	.22	.31	.26	.21		.42	—.22
[.14]	.21	.20	.43	.45	.32	.39	.31	.42		—.36
—.18	[—.12]	[—.11]	—.33	—.21	—.27	—.19	—.21	—.22	—.36	

TABLE IX

CRUDE COEFFICIENTS—WADLEIGH HIGH SCHOOL AND HORACE MANN SCHOOL

	Algebraic Computation	Matching Equations and Problems	Matching Nth Terms and Series	Interpolation	Missing Steps in Series	Inference with Symbols	Geometry
Algebraic Computation		.57	.30	.50	.59	.29	.42
Matching Equations and Problems	.57		.26	.29	.45	.25	.36
Matching Nth Terms and Series	.30	.26		.26	.29	.22	.19
Interpolation	.50	.29	.26		.57	.39	.40
Missing Steps in Series	.59	.45	.29	.57		.29	.31
Inference with Symbols	.29	.25	.22	.39	.29		.30
Geometry	.42	.36	.19	.40	.31	.30	
Superposition	.37	.29	.13	.33	.27	.24	.32
Symmetry	.17	.22	.12	.22	.22	.12	.29
Matching Solids and Surfaces	.25	.09	.03	.23	.22	.14	.37
Geometrical Definitions	.24	.25	.08	.26	.31	.29	.50
Reasoning	.31	.24	.12	.26	.25	.30	.32
Arithmetic Problems	.36	.34	.15	.43	.42	.27	.27
Mixed Relations	.26	.27	.01	.28	.30	.17	.33
Logical Opposites	.36	.23	.22	.29	.33	.24	.27
Trabue Language Scales	.23	.19	.16	.14	.21	.10	.28
Thorndike Reading Tests	.26	.33	.14	.27	.18	.26	.30
Age	—.37	—.25	—.15	—.44	—.33	—.34	—.34

These crude coefficients, however, do not tell us accurately the real amount of correlation that exists in the case of any two functions. Apart altogether from the error due to sampling, of whose size we can judge and in the case of which we can protect ourselves from false reasoning by estimating the probable error, there are other important sources of fallacy. Thus not only do our crude coefficients represent the correlations found in a very limited group of individuals, they are also merely such measures of correspondence as arise from two sets of observations obtained by methods of experimentation more or less imperfect. Errors of the latter kind, which can assume large proportions in psychological work, cannot be got rid of by increasing the number of individuals examined. They do not tend to balance each other in the case of correlations as happens in determining group averages. Their tendency is to reduce the size of the coefficients calculated towards zero. In order to eliminate their effect, Spearman [4] has proposed certain formulae, "based on the idea that the

[4] Spearman, C., The Proof and Measurement of Association Between Two Things, *Am. Jour. Psych.* XV: 88, and Correlation calculated from Faulty Data, *British Jour. Psych.* III: 271.

TABLE IX—*Continued*

	Superposition	Symmetry	Matching Solids and Surfaces	Geometrical Definition	Reasoning	Arithmetic Problems	Mixed Relations	Logical Opposites	Trabue Language Scales	Thorndike Reading Tests	Age
	.37	.17	.25	.24	.31	.04	.26	.36	.23	.26	—.37
	.29	.22	.09	.25	.24	.34	.27	.23	.19	.33	—.25
	.13	.12	.03	.08	.12	.15	.01	.22	.16	.14	—.15
	.33	.22	.23	.26	.26	.43	.28	.29	.14	.27	—.44
	.27	.22	.22	.31	.25	.42	.30	.33	.21	.18	—.33
	.24	.12	.14	.29	.30	.27	.17	.24	.10	.26	—.34
	.32	.29	.37	.50	.32	.27	.33	.27	.28	.30	—.34
		.61	.35	.32	.34	.31	.21	.12	.11	.11	—.13
	.61		.30	.31	.23	.29	.22	.27	.21	.21	—.06
	.35	.30		.39	.40	.20	.26	.18	.17	.15	—.09
	.32	.31	.39		.23	.22	.31	.31	.28	.32	—.26
	.34	.23	.40	.23		.28	.22	.24	.20	.31	—.18
	.31	.29	.20	.22	.28		.23	.23	.33	.23	—.28
	.21	.22	.26	.31	.22	.23		.43	.28	.35	—.13
	.12	.27	.18	.31	.24	.23	.43		.37	.39	—.17
	.11	.21	.17	.28	.20	.33	.28	.37		.50	—.14
	.11	.21	.15	.32	.31	.23	.35	.39	.50		—.23
	—.13	—.06	—.09	—.26	—.18	—.28	—.13	—.17	—.14	—.23	

size of these accidental errors can be measured by the size of the discrepancies between successive measurements of the same things." These formulae have been criticised adversely by several writers,[5] the most serious charge levelled against them being that their assumption that errors of observation are themselves uncorrelated is unwarranted. Spearman admits the justice of this criticism in the case of "variations of a regular and continuously progressive character," while insisting that there is besides these a host of "variations of a discontinuously shifting sort" that cannot be controlled, as the former may, which are due to accident, and which are most scientifically dealt with by a process of elimination comparable to the familiar methods of "smoothing curves" or "taking means." Udny Yule has demonstrated the existence of the attenuation of coefficients of correlation by errors of observation by a still simpler proof and has shown the assumptions on which Spearman's Correction formulae are based. Spearman[6]

[5] Pearson, Karl, *Biometrika*, III: 160, and Drapers' Company Research Memoirs, *Biometric Series*, IV, 1907. Brown, Wm., The Essentials of Mental Measurement, Cambridge, 1911, 83.

[6] Spearman, C., General Intelligence—Objectively Determined and Measured, *Am. Jour. Psych.*, XV: 257.

has pointed out clearly the conditions under which correction can legitimately be applied.

If the observed coefficient of correlation is less than twice the probable error, since there is no conclusive evidence of the existence of positive correspondence between the traits under investigation, correction is out of the question. Where, however, the observed correlation is substantially greater than the probable error, say four or five times its amount, correspondence being established, we are justified in using a reasonable method of correction in order to bring the attenuated measure nearer to its most probable true value.

In the present investigation the particular formula [7] used was the following:

$$r_{pq} = \frac{\sqrt{(r_{p_1q_2})\,(r_{p_2q_1})}}{\sqrt{(r_{p_1p_2})\,(r_{q_1q_2})}}$$

r_{pq}	here indicates the true correlation between two series of measures p and q of the facts A and B.
p_1 and p_2	are two independent measures of A.
q_1 and q_2	are two independent measures of B.
$r_{p_1q_2}$	is the correlation obtained from the first measure of A and the second measure of B.
$r_{p_2q_1}$	is the correlation obtained from the second measure of A and the first measure of B.
$r_{p_1p_2}$	is the correlation between the two measures of A.
$r_{q_1q_2}$	is the correlation between the two measures of B.

In Tables X and XI the corrected coefficients for the two groups are given in full, even in cases where the low correlation and the high probable error scarcely warrant the correction being made. Where a coefficient is absent from the table it signifies that the Correction formula could not be applied owing to one of the crude coefficients being zero or the two being of unlike sign. In Table XII the results derived from both groups are amalgamated, those of Horace Mann School receiving double weight on account of their superior reliability.

[7] Thorndike, E. L., Theory of Mental and Social Measurements, New York, 1913, 179.

The probable error of these corrected coefficients may be approximately determined by the use of the formula:[8]

$$\frac{\text{P. E. of Corrected Coefficient}}{\text{P. E. of Crude Coefficient}} = \frac{\text{Corrected Coefficient}}{\text{Crude Coefficient}}$$

Since it exceeds the probable error of the corresponding raw coefficient in proportion to the amount of correction made, it is left to the reader to infer the required increase in each case.

[8] Burt, C., Experimental Tests of General Intelligence, *British Jour. Psych.* III: 111.

TABLE X

Corrected Coefficients—Wadleigh High School

	Algebraic Computation	Matching Equations and Problems	Matching Nth Terms and Series	Interpolation	Missing Steps in Series	Inference with Symbols	Geometry
Algebraic Computation		.65	.15	.48	.72		.37
Matching Equations and Problems	.65		.22	.09	.42	.02	.31
Matching Nth Terms and Series	.15	.22		.11	.06	.14	.11
Interpolation	.48	.09	.11		.45	.54	.25
Missing Steps in Series	.72	.42	.06	.45			.00
Inference with Symbols	.	.02	.14	.54			.45
Geometry	.37	.31	.11	.25	.00	.45	
Superposition	.33	.25	.72	.32	.27	.31	.08
Symmetry	.	.21	.	.11	.05	.38	.29
Matching Solids and Surfaces	.46	.08	.25	.30	.43	.03	.34
Geometrical Definitions	.38	.15	.11	.	.27	.	.81
Reasoning	.44	.33	.	.63	.36	.61	.17
Arithmetic Problems	.05	.33	—.14	.56	.39	.38	.27
Mixed Relations	.08	.46	—.25	.23	.15	.33	.40
Logical Opposites	.40	.49	.	.33	.42	.72	.32
Trabue Language Scales	.54	.64	.18	.12	.21	.51	.52
Thorndike Reading Tests	.	.44	.06	.30	—.16	.65	.
Age	—.30	—.18	—.03	—.44	—.21	.	—.12

TABLE XI

Corrected Coefficients—Horace Mann School

	Algebraic Computation	Matching Equations and Problems	Matching Nth Terms and Series	Interpolation	Missing Steps in Series	Inference with Symbols	Geometry
Algebraic Computation		.91	.48	.65	.81	.58	.58
Matching Equations and Problems	.91		.41	.48	.82	.55	.64
Matching Nth Terms and Series	.48	.41		.38	.49	.29	.30
Interpolation	.65	.48	.38		.84	.54	.58
Missing Steps in Series	.81	.82	.49	.84		.58	.58
Inference with Symbols	.58	.55	.29	.54	.58		.44
Geometry	.58	.64	.30	.58	.58	.44	
Superposition	.51	.48	.10	.42	.40	.37	.59
Symmetry	.23	.30	.15	.29	.37	.09	.39
Matching Solids and Surfaces	.32	.14	—.01	.32	.33	.25	.62
Geometrical Definitions	.14	.45	.04	.37	.52	.43	.72
Reasoning	.43	.37	.24	.25	.38	.57	.59
Arithmetic Problems	.72	.64	.35	.63	.79	.47	.48
Mixed Relations	.43	.36	.09	.39	.51	.24	.49
Logical Opposites	.57	.67	.25	.42	.55	.32	.45
Trabue Language Scales	.34		.35	.22	.53		.53
Thorndike Reading Tests	.57	.62	.19	.42	.49	.41	.55
Age	—.49	—.38	—.20	—.49	—.48	—.51	—.50

TABLE X—*Continued*

Superposition	Symmetry	Matching Solids and Surfaces	Geometrical Definitions	Reasoning	Arithmetic Problems	Mixed Relations	Logical Opposites	Trabue Language Scales	Thorndike Reading Tests	Age
.33		.46	.38	.44	.05	.08	.40	.54		—.30
.25	.21	.08	.15	.33	.33	.46	.49	.64	.44	—.18
.72		.25	.11		—.14	—.25		.18	.06	—.03
.32	.11	.30		.63	.56	.23	.33	.12	.30	—.44
.27	.05	.43	.27	.36	.39	.15	.42	.21	—.16	—.21
.31	.38	.03		.61	.38	.33	.72	.51	.65	
.08	.29	.34	.81	.17	.27	.40	.32	.52		—.12
	.68	.67	.37	.47	.26	.28	.03			—.03
.68		.27	.69	.19	.41	.09		.20	.33	.06
.67	.27		.06	.59		.30	.06			
.37	.69	.06	.	.	.	.	.58	.60		
.47	.19	.59	.		.47	.32	.37	.38		—.15
.26	.41	.	.	.47		—.06		.41		—.26
.28	.09	.30	.	.32	—.06		.74	.07		
.63	.	.06	.58	.37	.	.74		.60	.47	—.08
.	.20	.	.60	.38	.41	.07	.60		.67	
.	.33	.		.	.		.47	.67	.	.06
—.03	.06	.		—.15	—.26	.	—.08		.06	

TABLE XI—*Continued*

Superposition	Symmetry	Matching Solids and Surfaces	Geometrical Definitions	Reasoning	Arithmetic Problems	Mixed Relations	Logical Opposites	Trabue Language Scales	Thorndike Reading Tests	Age
.51	.23	32	.14	.43	.72	.43	.57	.34	.57	—.49
.48	.30	.14	.45	.37	.64	.36	.67		.62	—.38
.10	.15	—.01	.04	.24	.35	.09	.25	.35	.19	—.20
.42	.29	.32	.37	.25	.63	.39	.42	.22	.42	—.49
.40	.37	.33	.52	.38	.79	.51	.55	.53	.49	—.48
.37	.09	.25	.43	.57	.47	.24	.32		.41	—.51
.59	.39	.62	.72	.59	.48	.49	.45	.53	.55	—.50
	.68	.46	.46	.48	.53	.26	.21	.28	.06	—.20
68		.40	.31	.08	.38	.30	.34	.36	.25	—.12
.46	.40		.63	.67	.33	.45	.32	.46	.31	—.13
.46	.31	.63		.44	.46	.49	.28	.54	.61	—.36
.48	.08	.67	.44		.44	.34	.36	.45	.69	—.24
.53	.38	.33	.46	.44		.55	.47	.59	.49	—.41
.26	.30	.45	.49	.34	.55		.37	.52	.58	—.22
.21	.34	.32	.28	.36	.47	.37		.45	.54	—.27
.28	.36	.46	.54	.45	.59	.52	.45		.96	—.33
.06	.25	.31	.61	.69	.49	.58	.54	.96		—.46
—.20	—.12	—.13	—.36	—.24	—.41	—.22	—.27	—.33	—.46	

TABLE XII

CORRECTED COEFFICIENTS—WADLEIGH HIGH SCHOOL AND HORACE MANN SCHOOL

	Algebraic Computation	Matching Equations and Problems	Matching Nth Terms and Series	Interpolation	Missing Steps in Series	Inference with Symbols	Geometry
Algebraic Computation		.82	.37	.60	.78	.35	.51
Matching Equations and Problems	.82		.35	.35	.68	.37	.53
Matching Nth Terms and Series	.37	.35		.29	.35	.24	.23
Interpolation	.60	.35	.29		.71	.54	.47
Missing Steps in Series	.78	.68	.35	.71		.39	.39
Inference with Symbols	.35	.37	.24	.54	.39		.45
Geometry	.51	.53	.23	.47	.39	.45	
Superposition	.45	.40	.31	.39	.35	.35	.42
Symmetry	.19	.27	.10	.23	.26	.19	.35
Matching Solids and Surfaces	.37	.12	.09	.31	.36	.18	.53
Geometrical Definitions	.22	.35	.06	.31	.44	.29	.75
Reasoning	.44	.35	.16	.38	.37	.58	.45
Arithmetic Problems	.50	.53	.19	.61	.66	.44	.41
Mixed Relations	.32	.39	—.02	.34	.39	.27	.46
Logical Opposites	.51	.61	.17	.39	.50	.46	.41
Trabue Language Scales	.41	.70	.29	.18	.42	.40	.53
Thorndike Reading Tests	.38	.56	.15	.38	.33	.49	.37
Age	—.43	—.31	—.15	—.47	—.39	—.34	—.38

TABLE XII—*Continued*

Superposition	Symmetry	Matching Solids and Surfaces	Geometrical Definitions	Reasoning	Arithmetic Problems	Mixed Relations	Logical Opposites	Trabue Language Scales	Thorndike Reading Tests	Age
.45	.19	.37	.22	.44	.50	.32	.51	.41	.38	—.43
.40	.27	.12	.35	.35	.53	.39	.61	.70	.56	—.31
.31	.10	.09	.06	.16	.19	—.02	.17	.29	.15	—.15
.39	23	.31	.31	.38	.61	.34	.39	.18	.38	—.47
.35	.26	.36	.44	.37	.66	.39	.50	.42	.33	—.39
.35	.19	.18	.29	.58	.44	.27	.46	.40	.49	—.34
.42	.35	.53	.75	.45	.41	.46	.41	.53	.37	—.38
	.68	.53	.43	.48	.44	.27	.15	.19	.04	—.14
.68		.36	.44	.12	.39	.23	.23	.31	.28	—.10
.53	.36		.44	.64	.22	.40	.24	.50	.21	—.09
.43	.44	.44		.29	.31	.33	.38	.56	.41	—.24
.48	.12	.64	.29		.45	.33	.36	.42	.46	—.21
.44	.39	.22	.31	.45		.35	.31	.53	.33	—.32
.27	.23	.40	.33	.33	.35		.49	.37	.45	—.14
.15	.23	.24	.38	.36	.31	.49		.50	.51	—.21
.18	.31	.50	.56	.42	.53	.37	.50	.	.87	—.22
.04	.28	.21	.41	.46	.33	.39	.51	.87		—.29
—.14	—.10	—.09	—.24	—.21	—.32	—.14	—.21	—.22	—.29	

We can now turn these statistical results to account in analyzing the degree and the kind of interdependence that exists between the various mental functions measured. Certain general features characterizing all the Correlation tables are worthy of note. In spite of the complexity of the interrelations which they show, theoretical conclusions of value can be deduced from a rapid survey. There is apparent a tendency for allied tests to correlate together more closely than those from different groups. Thus coefficients derived from pairs of tests of algebraic abilities are in general higher than those that ensue from combining two tests, one of algebraic and the other of geometrical capacities. An analysis of the tables yields the following interesting results.

In Table VII ten of the coefficients of correlation between mathematical functions exceed .50 and of these every single pair of tests belong to the same group of abilities. In the same table .26 of the coefficients of correlation between mathematical capacities are greater than .40 and of these twenty-one are derived from tests of the same class. Similarly, in Table VIII, of 26 coefficients exceeding .40 in amount, twenty-one are obtained from tests of allied abilities.

The results are as marked in the case of the corrected coefficients. In Table X, of eleven coefficients over .50, ten issue from pairs of functions both of which belong either to the group of algebraic abilities or to that of geometrical abilities, and similarly in Table XI, of twenty-three coefficients exceeding .55, nineteen result from tests of the same general kind.

The combination of the Wadleigh and the Horace Mann coefficients, both raw and corrected, in Tables IX and XII, offers equally striking evidence.

This tendency can be generally discerned in the changes in the magnitude of the correlations, as we pass from the tests belonging to one field to those of another. It is perhaps most obvious in the case of the somewhat specialized tests involving intuitive grasp of spatial relations.

Another feature common to all the Correlation tables is the absence of negative coefficients with the one exception of the almost universal negative correlation of every function tested with

age. Apart from the latter, each negative coefficient found is less than twice its probable error. Consequently it may be due to accidental flaws in the method of measurement and cannot be regarded as demonstrating the presence of inverse relation. Practically all such are cases of absence of correspondence between the traits measured. Even for tasks apparently so dissimilar as the solving of arithmetic problems, the interpolation of numbers and the superposition of geometrical figures, a positive correspondence exists, though frequently it is small in amount.

There is an obvious tendency for age to correlate inversely with the functions measured. Thus it is apparent that the tests are indicative of the qualities which cause a pupil to begin the study of mathematics young, or to progress through school rapidly, or both.

The Correlation tables give a partial analysis of mathematical intelligence, expressing in precise quantitative form the kind and amount of kinship between the various abilities examined. For deeper insight into the nature of these traits further statistical treatment is necessary. Up to this point we have considered them in isolation. We must now pass to the inquiry into the relative status of each in mathematical intelligence. At the same time we shall also determine the characteristics in a test which produce high correlation with mathematical ability, and discover, if possible, the common psychological factor or factors which explain the manifold correspondences that appear in the tables.

For this purpose we require a measure of mathematical ability with reference to which we can determine the value of each test. We might obtain such a measure in a variety of ways. It is desirable, for example, that we should have an independent estimate of the efficiency in algebra, geometry, and arithmetic of the pupils examined. This was in fact obtained from the school marks in these subjects, which they had received up to date.

A second measure that might be used as an index to general proficiency in mathematical work is the grand total of the scores in the tests of all of the functions which can lay substantial claim to be called mathematical.

A third possible method of determining the relative value of the tests as measures of mathematical ability is by comparison of

the average of any particular test's correlations with all the others with their corresponding averages.[9]

Each of these standards was used in this study. In the case of the first and third, the procedure is simple and straightforward, requiring no explanation. It is necessary, however, to give some account of how our second standard was derived.

The difficulties in the way of combining the results of several tests are the incommensurability of certain measures (some being in terms of time and others in terms of accuracy), the different averages of the same group in different tests, and the different variabilities of the same group in different tests. To eliminate both the absolute value of the average and that of the variability Woodworth[10] has suggested that we let the average be counted as zero, so that the standing of each individual is expressed as a deviation, and to make the measure of variability the unit deviation, so that all deviations are expressed as fractions or multiples of this unit. Thus each individual in each test is assigned a position in the distribution of the group. He stands above or below the group average and so much above or below as compared with the average variation of the group. Thus having determined each individual's position with reference to the central tendency in the case of each test, these values are reduced to suitable proportions to one another. If we desire all the tests combined to have an equal influence in the composite, we must multiply the deviations by such factors as will make their variabilities equal. Where, however, we wish for any reason to attach greater weight to certain tests than others we must multiply the deviations of the tests in question by such factors as will make their variability greater in the required proportion.

It is certainly desirable in constructing a composite measure of the individual's achievement in the several tests that we should take cognizance of the factors exercising a significant influence upon the relations to be investigated. Obviously these may be

[9] Compare McCall, W. A., Correlation of Some Psychological and Educational Measurements, Columbia University, Contributions to Education, Teachers College Series, No. 79, 35.

[10] Woodworth, R. S., Combining the Results of Several Tests, *Psychological Review*, XIX: 97 and Yule, G. Udny, An Introduction to the Theory of Statistics, 218-219, ¶ 12, Correlation due to Heterogeneity of Material, London, 1916.

numerous, but some we shall have to neglect because we lack the knowledge necessary to decide what importance to attach to them. The tests certainly differ in their value as indices to mathematical ability and this would have to be recognized in a perfect measure of general mathematical efficiency. The best weights to attach to each test might be determined by the method of partial correlation coefficients, which has been devised by Edgeworth,[11] Pearson [12] and Yule [13] and developed by Kelley.[14] The method, however, becomes exceedingly cumbrous and the labor it involves enormous, where the variables are at all numerous, and probably sufficiently satisfactory results can be had, where almost any reasonable weighting is used.

The latter empirical method of arriving at a best possible composite for mathematical ability was the one followed in this study. The measures for each function were weighted with reference to two main factors, the importance of the ability measured and the reliability of the test from which they were derived. In accordance with these principles six of the tests were given double weight in the composite developed. These were Algebraic Computation, Matching Equations and Problems, Geometry Test, Matching Solids and Surfaces, Interpolation, and Arithmetic Problems. It will be seen that these include the most reliable tests and, as far as we can judge, the most representative tests of the total number.

In Tables XIII and XIV are given the data upon which the new measures of mathematical ability compounded from all the tests were based.

[11] Edgeworth, F. Y., On Correlated Averages, *Phil. Mag.* 5th Series, XXXIV: 194.

[12] Pearson, Karl, Regression, Heredity, and Panmixia, *Phil.* Trans. Roy. Soc., Series A, CLXXX: 253.

[13] Yule, G. Udny, On the Theory of Correlation for any Number of Variables Treated by a new System of Notation, Proc. Roy. Soc., Series A, LXXIX: 182.

[14] Kelley, T. L., Educational Guidance, Columbia University, Contributions to Education, Teachers College Series, No. 71, 1914, and Tables for Facilitating the Calculation of Partial Coefficients of Correlation, etc., *Univ. of Texas Bulletin*, 1916, No. 27.

TABLE XIII

WEIGHTS GIVEN TO THE TESTS INCLUDED IN THE COMPOSITE FOR MATHEMATICAL ABILITY
WADLEIGH HIGH SCHOOL

		Standard Deviation	*Average Standard Deviation*	*Desired Weight*	*Multiple*
Algebraic Computation	(1)	7.03	6.80	2	3
	(2)	6.57			
Matching Equations and Problems	(1)	3.13	3.34	2	8
	(2)	3.55			
Matching *N*th Terms and Series..	(1)	4.33	3.97	1	3
	(2)	3.62			
Interpolation	(1)	8.14	6.34	2	4
	(2)	4.55			
Missing Steps in Series..........	(1)	3.03	3.21	1	4
	(2)	3.39			
Inference with Symbols	(1)	1.73	1.73	1	6
	(2)	1.74			
Geometry	(1)	5.25	5.12	2	5
	(2)	4.99			
Superposition	(1)	5.69	5.95	1	2
	(2)	6.22			
Symmetry	(1)	5.43	5.95	1	2
	(2)	6.55			
Geometrical Definitions	(1)	5.58	5.21	1	2
	(2)	4.85			

Matching Solids and Surfaces....	(1)	6.19	5.92	2	4
	(2)	5.65			
Reasoning	(1)	3.89	3.63	1	3
	(2)	3.37			
Arithmetic Problems	(1)	1.36	1.31	2	20
	(2)	1.27			

Multiple equals the number by which the deviations of the test concerned were multiplied, in order to give it the desired weight.

TABLE XIV

HORACE MANN SCHOOL

		Standard Deviation	*Average Standard Deviation*	*Desired Weight*	*Multiple*
Algebraic Computation	(1)	6.29	6.78	2	6
	(2)	7.28			
Matching Equations and Problems	(1)	4.95	6.31	2	6
	(2)	7.67			
Matching *N*th Terms and Series..	(1)	5.28	5.77	1	4
	(2)	6.26			
Interpolation	(1)	37.77	39.48	2	1
	(2)	41.20			
Missing Steps in Series	(1)	2.15	2.24	1	9
	(2)	2.34			
Inference with Symbols	(1)	11.48	11.55	1	2
	(2)	11.62			
Geometry	(1)	5.00	5.14	2	8
	(2)	5.29			

Superposition	(1)	5.82	6.60	1	3
	(2)	7.39			
Symmetry	(1)	8.69	9.18	1	2
	(2)	9.68			
Geometrical Definitions	(1)	8.06	8.04	2	5
	(2)	8.03			
Matching Solids and Surfaces	(1)	6.02	5.96	1	3
	(2)	5.91			
Reasoning	(1)	3.78	3.50	1	6
	(2)	3.22			
Arithmetic Problems	(1)	1.37	1.36	2	30
	(2)	1.35			

Multiple equals the number by which the deviations of the test concerned were multiplied, in order to give it the desired weight.

The coefficients of reliability for the composites for the two groups were calculated in the usual manner and are given in Tables XV and XVI along with the reliability coefficients for composites of algebraic, geometrical and verbal ability, which were also made on similar lines for use in another connection.

TABLES XV AND XVI

Reliability Coefficient for Each Composite and for its Two Applications Combined

	Wadleigh High School		Horace Mann School	
	$r1$	$r2$	$r1$	$r2$
Mathematical ability	.86	.92	.93	.96
Algebraic ability	.78	.88	.93	.96
Geometrical ability	.76	.86	.89	.94
Verbal ability	.71	.83	.75	.86

By means of this standard we can now ascertain the order of correlation of each test with mathematical capacity and so determine the relative worth of each test as a measure of that ability. The results of our calculation are presented in Tables XVII, XVIII, XIX, XX, XXI, XXII, which give the values of the coefficients both crude and corrected for each group separately and for the two groups combined. The crude coefficients are more significant for practical diagnosis, but their corrected values probably give a more true measure of the amount of correspondence that exists between each function tested and mathematical ability.

TABLE XVII

Coefficients of Correlation Between Each Test with the Composite for Mathematical Ability Arranged in Order of Magnitude (Crude)
Wadleigh High School

	r
Mathematical Tests	
Algebraic Computation	.55
Interpolation	.55
Superposition	.52
Missing Steps in Series	.49
Geometry	.49
Matching Equations and Problems	.49
Symmetry	.45
Reasoning	.41
Arithmetic Problems	.39
Matching Solids and Surfaces	.38
Geometrical Definitions	.30
Matching *N*th Terms and Series	.23
Inference with Symbols	.22
Verbal Ability Tests	
Trabue Language Scales	.39
Logical Opposites	.38
Mixed Relations	.23
Thorndike Reading Scales	.22

TABLE XVIII

Coefficients of Correlation Between Each Test with the Composite for Mathematical Ability Arranged in Order of Magnitude (Crude)

Horace Mann School

	r
Mathematical Tests	
Algebraic Computation	.76
Interpolation	.72
Missing Steps in Series	.70
Geometry	.69
Arithmetic Problems	.61
Matching Equations and Problems	.61
Geometrical Definitions	.60
Superposition	.60
Inference with Symbols	.58
Reasoning	.53
Matching Solids and Surfaces	.48
Symmetry	.48
Matching *N*th Terms and Series	.41
Verbal Ability Tests	
Reading—Understanding of Sentences	.47
Mixed Relations	.47
Logical Opposites	.42
Trabue Language Scales	.33

TABLE XIX

Wadleigh High School and Horace Mann School (Combined)

	r	P.E.
Mathematical Tests		
Algebraic Computation	.69	.05
Interpolation	.66	.04
Missing Steps in Series	.63	.06
Geometry	.63	.05
Superposition	.57	.02
Matching Equations and Problems	.57	.03
Arithmetic Problems	.54	.06
Geometrical Definitions	.50	.07
Reasoning	.49	.03
Symmetry	.47	.07
Inference with Symbols	.46	.09
Matching Solids and Surfaces	.45	.03
Matching *N*th Terms and Series	.35	.04
Verbal Ability Tests		
Logical Opposites	.41	.01
Thorndike Reading Scales	.39	.06
Mixed Relations	.39	.06
Trabue Language Scales	.35	.01

TABLE XX

COEFFICIENTS OF CORRELATION BETWEEN EACH TEST WITH THE COMPOSITE FOR MATHEMATICAL ABILITY ARRANGED IN ORDER OF MAGNITUDE (CORRECTED)

WADLEIGH HIGH SCHOOL

	r
Mathematical Tests	
Interpolation	.70
Matching Solids and Surfaces	.68
Reasoning	.68
Algebraic Computation	.67
Matching Equations and Problems	.66
Geometry	.64
Arithmetic Problems	.62
Superposition	.62
Missing Steps in Series	.61
Geometrical Definitions	.53
Symmetry	.50
Inference with Symbols	.47
Matching *N*th Terms and Series	.31
Verbal Ability Tests	
Trabue Language Scales	.62
Logical Opposites	.51
Thorndike Reading Scales	.32
Mixed Relations	.23

TABLE XXI

HORACE MANN SCHOOL

	r
Mathematical Tests	
Algebraic Computation	.87
Missing Steps in Series	.87
Geometry	.83
Matching Equations and Problems	.81
Arithmetic Problems	.80
Interpolation	.77
Superposition	.68
Geometrical Definitions	.68
Inference with Symbols	.66
Reasoning	.64
Matching Solids and Surfaces	.59
Symmetry	.51
Matching *N*th Terms and Series	.46

TABLE XXI—*Continued*

Verbal Ability Tests

Reading—Understanding of Sentences	.63
Logical Opposites	.57
Mixed Relations	.55
Trabue Language Scales	.54

TABLE XXII

COEFFICIENTS OF CORRELATION BETWEEN EACH TEST WITH THE COMPOSITE FOR MATHEMATICAL ABILITY ARRANGED IN ORDER OF MAGNITUDE (CORRECTED)

WADLEIGH HIGH SCHOOL AND HORACE MANN SCHOOL (COMBINED)

Mathematical Tests	*r*	P.E.
Algebraic Computation	.81	.05
Missing Steps in Series	.78	.06
Geometry	.76	.05
Matching Equations and Problems	.76	.04
Interpolation	.75	.02
Arithmetic Problems	.74	.04
Superposition	.66	.01
Reasoning	.65	.01
Geometrical Definitions	.63	.04
Matching Solids and Surfaces	.62	.07
Inference with Symbols	.60	.05
Symmetry	.51	.03
Matching *N*th Terms and Series	.41	.04
Verbal Ability Tests		
Trabue Language Scales	.57	.02
Logical Opposites	.55	.01
Thorndike Reading Scales	.53	.08
Mixed Relations	.44	.08

The uniformity in the results obtained will be readily recognized. When the observed quantities of correlation for the various tests are compared with their probable error, such differences in rank as are found between the two applications appear negligi-

ble. It will be profitable at this point to compare with the above results the relative positions of the tests, when arranged in the order of the magnitude of their correlation with mathematical ability, using our third standard. These have been computed from the amalgamated results of the two groups for both crude and corrected coefficients. They are included in Table XXIII.

TABLE XXIII

AVERAGE CORRELATION OF EACH TEST WITH EVERY TEST ARRANGED IN ORDER OF MAGNITUDE: WADLEIGH HIGH SCHOOL AND HORACE MANN SCHOOL COMBINED

Crude Coefficients:

	r
Algebraic Computation	.363
Missing Steps in Series	.349
Geometry	.346
Interpolation	.345
Superposition	.323
Matching Equations and Problems	.300
Geometrical Definitions	.285
Reasoning	.273
Arithmetic Problems	.268
Symmetry	.258
Inference with Symbols	.250
Matching Solids and Surfaces	.249
Matching *N*th Terms and Series	.195

Corrected Coefficients:

Missing Steps in Series	.478
Algebraic Computation	.466
Geometry	.457
Superposition	.435
Interpolation	.432
Arithmetic Problems	.429
Matching Equations and Problems	.426
Reasoning	.392
Inference with Symbols	.364
Geometrical Definitions	.360
Matching Solids and Surfaces	.345
Symmetry	.298
Matching *N*th Terms and Series	.228

The order of the tests corresponds closely to that obtained from the application of the second standard, but the differences in the coefficients are extremely small and suggest a chance distribution. On the other hand, by means of the first standard used we can decide between the tests as more or less indicative of mathematical intelligence. Thus of the thirteen tests of algebraic and geometrical abilities, seven yield coefficients (Table XXII) above .65, while six give values below .65. Obviously the former tests probe certain characteristics more fundamental in mathematical efficiency than the latter. It is equally clear, however, that no single test is a sufficient index to mathematical mastery. Even when the coefficients have been corrected for accidental errors of measurement, no test correlates perfectly with the composite. When it is remembered that the latter includes these tests now correlated with it, this fact will be all the more striking.[15] Among the highest observed raw coefficients for the two groups combined is that between the composite for mathematical ability and Algebraic Computation.[16] Its value is only .69. Moreover, the Symmetry test [17] (to take only one instance), with which apparently it has little observable relationship (.17) correlates with the composite [18] to the extent of .47. The corresponding corrected coefficients [19], [20], [21] are .81, .19, and .51. It seems that mathematical ability is a complex resultant of many loosely knit capacities, all working together.

Further light is thrown upon its nature by a consideration of the coefficients obtained in the case of the tests of verbal ability. These tests, it has to be remembered, unlike the mathematical tests, were not included in the composite. Their value is notably high. To make one or two comparisons from the corrected and combined coefficients of Wadleigh and Horace Mann, greater kinship apparently exists between mathematical ability and the

[15] The relationship of each test and the composite independent of its own contribution to the latter could be determined by the use of Partial Coefficients of Correlation. Each test, however, measures the efficiency of an activity, which has a prima facie claim to be called mathematical and therefore its influence in the composite ought to be expressed in the coefficient denoting the correspondence between mathematical ability and the function measured by it. The positive correspondence due to this is not properly described as spurious correlation.

[16] See Table XIX.

[17] See Table IX.

[18] See Table XIX.

[19] See Table XXII.

[20] See Table XII.

[21] See Table XXII.

functions measured by the Trabue Language Scale, the Thorndike Reading Tests and the test of Logical Opposites than between the former and the functions measured by the Matching *N*th Terms and Series or the Symmetry tests. (See Tables XIX and XXII.)

Other instances of the influence of ability with words upon ability in algebra and geometry can be had from an analysis of the tests which involve similar mental acts, but differ as regards content. Such a group of tests are the Interpolation test, the Missing Steps in Series test and the Trabue Language Scales. Each of these demands an anlysis of the given facts and their supplementation or completion in such a way as to produce a coherent and inclusive whole. The differences lie merely in the material. In the first and second of these the data are numbers; in the third, the data are words. The correspondences found between these functions show how important a rôle language plays in these performances. Between the Interpolation test and the Missing Steps in Series test the coefficient of correlation amounted to .57 (crude) [22] while between the Interpolation test and the Trabue Language Scales it was only .14 (crude).[22] Similarly between the Missing Steps in Series test and the Trabue Language Scales the correlation observed was .21 (crude).[22] The corresponding coefficients, when corrected, became .71, .18, and .42 respectively.[23]

Again Matching Problems and Equations, Matching *N*th Terms and Series and Matching Solids and Surfaces demand similar mental processes of analysis and identification, while differing considerably in content. In the first pair (one involving words and numbers, and the other involving numbers only) the correspondence amounted to .26,[24] while between Matching Problems and Matching Solids and Surfaces and between Matching *N*th Terms and Matching Solids and Surfaces,[25] there was .09 and .03 respectively. The corresponding corrected coefficients [26] were .35, .12, and .09.

More striking still is the evidence of the complexity of mathematical capacity, which follows from grouping the tests of kindred nature. When a composite of algebraic ability is made in the

[22] See Table IX.
[23] See Table XII. [24] See Table IX. [25] *Ibid.* [26] See Table XII.

same way as described earlier in this study, weights being given in identical proportions to the various tests and when corresponding composites are made for geometrical ability and verbal or language ability, results of considerable interest are obtained.

Before presenting these, however, the method in which the composite for verbal ability was constructed will be shown. In Tables XXIV and XXV the weights given to the tests are indicated. The reliability coefficients for these new composites were, in the case of the Wadleigh High School,[27] .78, .76 and .71 for algebraic, geometrical and verbal ability, and similarly for Horace Mann School,[28] they were .93, .89 and .75 respectively.

Tables XXVI, XXVII and XXVIII present the various coefficients of correlation between these composites both crude and corrected, and separately and combined.

TABLE XXIV

WEIGHTS GIVEN TO EACH TEST IN THE COMPOSITE FOR VERBAL ABILITY
WADLEIGH HIGH SCHOOL

		Standard Deviation	*Average Standard Deviation*	*Desired Weight*	*Multiple*
Mixed Relations	(1)	5.09	4.74	1	3
	(2)	4.39			
Logical Opposites	(1)	7.78	6.97	1	2
	(2)	6.17			
Trabue Language Scales	(1)	2.62	2.84	2	10
	(2)	3.07			
Thorndike Reading Scales	(1)	2.91	3.56	2	8
	(2)	4.22			

Multiple equals the number by which the deviations of the test concerned were multiplied, in order to give it the desired weight.

[27] See Table XV.
[28] See Table XVI.

TABLE XXV

WEIGHTS GIVEN TO EACH TEST IN THE COMPOSITE FOR VERBAL ABILITY
HORACE MANN SCHOOL

		Standard Deviation	Average Standard Deviation	Desired Weight	Multiple[1]
Mixed Relations	(1)	9.62	9.16	1	2
	(2)	8.71			
Logical Opposites	(1)	19.39	21.20	1	1
	(2)	23.02			
Trabue Language Scales	(1)	3.20	3.64	2	12
	(2)	4.09			
Thorndike Reading Scales	(1)	2.77	5.49	2	8
	(2)	8.22			

[1] Multiple equals the number by which the deviations of the test concerned were multiplied, in order to give it the desired weight.

TABLE XXVI

COEFFICIENTS OF CORRELATION BETWEEN THE COMPOSITES FOR MATHEMATICAL ABILITY, ALGEBRAIC ABILITY, GEOMETRICAL ABILITY AND VERBAL ABILITY
WADLEIGH HIGH SCHOOL

CRUDE

	r
Mathematical ability and Verbal ability.................	.44
Algebraic ability and Geometrical ability................	.38
Algebraic ability and Verbal ability.....................	.42
Geometrical ability and Verbal ability..................	.41

CORRECTED

	r
Mathematical ability and Verbal ability.................	.57
Algebraic ability and Geometrical ability................	.49
Algebraic ability and Verbal ability.....................	.56
Geometrical ability and Verbal ability..................	.56

TABLE XXVII

Coefficients of Correlation Between the Composites for Mathematical Ability, Algebraic Ability, Geometrical Ability and Verbal Ability

Horace Mann School

Crude

	r
Mathematical ability and Verbal ability	.58
Algebraic ability and Geometrical ability	.52
Algebraic ability and Verbal ability	.48
Geometrical ability and Verbal ability	.49

Corrected

	r
Mathematical ability and Verbal ability	.69
Algebraic ability and Geometrical ability	.57
Algebraic ability and Verbal ability	.57
Geometrical ability and Verbal ability	.61

TABLE XXVIII

Coefficients of Correlation Between the Composites for Mathematical Ability, Algebraic Ability, Geometrical Ability and Verbal Ability

Wadleigh High School and Horace Mann School (Combined)

Crude

	r	P.E.
Mathematical ability and Verbal ability	.54	.01
Algebraic ability and Verbal ability	.46	.01
Geometrical ability and Verbal ability	.47	.07
Algebraic ability and Geometrical ability	.47	.03

Corrected

	r	P.E.
Mathematical ability and Verbal ability	.65	.09
Algebraic ability and Verbal ability	.57	.05
Geometrical ability and Verbal ability	.59	.01
Geometrical ability and Algebraic ability	.54	.02

The closeness of relationship between mathematical ability and ability with words as represented in these four tests, Mixed Relations, Logical Opposites, Trabue Language Scales, and the Thorndike Reading Tests is important. Several writers have pointed out, notably Suzzallo,[29] that in primary arithmetic the problem of teaching children to reason is largely a matter of teaching children language and how to use it. "Reasoning in school problems has far more to do with the language involved in a problem than with the numbers or combinations of numbers." It would appear that this still remains true in later mathematical work. The ability to understand sentences, to conceive clearly the meaning of a given problem, is as important an element in its solution as any connection it has with algebraic symbols or their manipulation. The relative significance of ability with language is made apparent by the degree of correlation that these coefficients reveal.

It would thus appear that the tests in verbal ability enable us to prophesy efficiency in algebra with as great a chance of success as the tests in geometry would and contrariwise that they are as reliable indices of the characteristics which make a successful student of geometry, as the algebra tests are.

These results raise the important question of what the true relationship between algebraic ability and geometrical ability is, when these are freed from this common factor, verbal ability. How far is the correlation between algebraic and geometrical ability due to the correlation that exists between each of these and the abilities measured by the tests of what we have called verbal ability and to what extent is it independent of the latter?

To find an answer to this, recourse was had to the method of Partial Coefficients of Correlation, by which the relationship between two functions for a constant value of a third can be determined.

[29] Suzzallo, H., Reasoning in Primary Arithmetic, *California Education*, June, 1906: 189.

The formula used was the following: [30]

$$r_{12\cdot 3} = \frac{r_{12} - r_{13}\, r_{23}}{\sqrt{(1 - r^2_{1\cdot 3})\,(1 - r^2_{2\cdot 3})}}$$

in which $r_{12\cdot 3}$ indicates the correlation between traits 1 and 2, for a constant value of trait 3. The reasoning underlying the Partial Correlation formula for three variables can be simply illustrated. Suppose that of the sixty-one Horace Mann students examined, ten are of approximately equal capacity in the verbal ability tests. The achievements in algebra and geometry of this group, in which verbal ability is constant, are then correlated. The resulting coefficient gives the partial correlation of algebraic and geometrical abilities for a constant value of verbal ability; that is, it expresses the relation of ability in algebra to ability in geometry independent of language ability; or, in other words, it represents the extent to which these abilities are related apart from their common connection with the ability to deal with words.

The application of this method gave significant results. Let us consider the crude coefficients. For the one group tested the new relationship between algebraic ability and geometrical ability finds expression in the coefficient .25 ± .07. The corresponding result for the other group is .37 ± .05. Averaging these coefficients and attaching double weight to the Horace Mann figures in accordance with our usual procedure, the value of the coefficient of correspondence between the two abilities obtained for the total number of persons tested is .33 ± .03.

Undoubtedly these partial coefficients do not represent the pure relationship between algebraic ability and geometrical ability in isolation from the influence of their mutual relations with all other traits. With the elimination of the latter there would be still further reduction in the coefficient. It should be remembered also that irrelevant factors such as age certainly affect the degree of correspondence observed. The correlations of the various composites with age and their interrelations after the effect of age

[30] Yule, G. Udny, An Introduction to the Theory of Statistics, London, 1916 Chapter 12.

has been eliminated are interesting in this connection. These are summarized in Tables XXIX, XXX, and XXXI.

TABLE XXIX

COEFFICIENTS OF CORRELATION BETWEEN THE COMPOSITES FOR MATHEMATICAL ABILITY, ALGEBRAIC ABILITY, GEOMETRICAL ABILITY, AND VERBAL ABILITY AND AGE, FOR WADLEIGH HIGH SCHOOL, HORACE MANN SCHOOL AND BOTH COMBINED

CRUDE COEFFICIENTS

	W.H.S.	H.M.S.	W.H.S. & H.M.S.
Mathematical ability and Age..	—.30	—.48	—.42
Algebraic ability and Age.....	—.36	—.52	—.47
Geometrical ability and Age..	[—.04]	—.33	—.21
Verbal ability and Age.......	[—.07]	—.35	—.26
Diagnostic composite and Age	—.27	—.52	—.44

Coefficients less than 2 P.E. are put in square brackets.

TABLE XXX

CORRECTED COEFFICIENTS

	W.H.S.	H.M.S.	W.H.S. & H.M.S.
Mathematical ability and Age..	—.32	—.50	—.44
Algebraic ability and Age.....	—.41	—.53	—.49
Geometrical ability and Age..		—.35	
Verbal ability and Age.......	—.03	—.40	—.27
Diagnostic composite and Age	—.29	—.55	—.46

TABLE XXXI

COEFFICIENTS OF CORRELATION BETWEEN THE COMPOSITES FOR MATHEMATICAL ABILITY, ALGEBRAIC ABILITY, GEOMETRICAL ABILITY AND VERBAL ABILITY, THE EFFECT OF AGE BEING ELIMINATED (CRUDE)

WADLEIGH HIGH SCHOOL, HORACE MANN SCHOOL AND BOTH COMBINED

	W.H.S.	H.M.S.	W.HS. & H.M.S.
Mathematical ability and Verbal ability.....	.44	.50	.48
Algebraic ability and Geometrical ability...	.39	.43	.42
Algebraic ability and Verbal ability........	.43	.37	.39
Geometrical ability and Verbal ability......	.41	.42	.42

By the application of the same method of Partial Coefficients of Correlation it is possible to ascertain the relation between algebraic ability and geometrical ability independent of ability with words and unaffected by age. In the case of the Wadleigh High School students the coefficient determined from the crude values was .26, for the Horace Mann students it was .33 and for these combined in the usual manner it was .30. Our results thus confirm those obtained by Brown [31] that algebra and geometry demand activities of different kinds, although algebraic and geometrical abilities are positively related as is usual in the case of desirable traits. They lend no support to the view that there is a "special capacity or faculty underlying mathematical ability, distinct from and with no essentially close connections with other forms of mental capacity."

The results so far obtained lend further support to the view that mathematical intelligence is complex in character, embracing a variety of mental processes, which are somewhat loosely related, but equally indispensable for successful accomplishment in the subject. A more penetrating analysis of its general nature has still to be made. Can our results afford any explanation of such correspondences as have been found? Can they suggest the characteristics in the tests which make for high correlation with mathematical ability? Is there any common psychological factor in those tests which correlate closely with mathematical ability? Are they saturated, as it were, with some quality which pervades them in different amounts? Does some feature common to the Algebraic Computation, Interpolation, Missing Steps in Series and the Geometry Tests explain why these tests correlate more closely with the composite than do the remaining tests?

We need not consider at this time such effects as that of general technique of administration, which undoubtedly exercises an important influence upon the correlations observed. For example, the mere duration of the time of testing (the lengthiness or shortness of the test) has a marked effect. Where the tests are supposedly given equal weight in the composite, in fact each is given a weight in rough proportion to the time of testing. When we review several of the explanations that might be given in answer

[31] See Brown, William. An Objective Study of Mathematical Intelligence. *Biometrika*, VII: 361.

to the questions propounded, such as the amount of novel or familiar experience the test entails, or the degree of complexity or simplicity it involves, or the demands it makes upon the ability to abstract and analyze and think selectively, we are led to the conclusion that while all of these, together with many other factors, may be determinants of the correlations found between mathematical ability and the tests, yet the highest common psychological factor, which explains the character of the correspondences revealed, is this ability to react to partial elements in a situation rather than to gross totals.[32] While mathematical intelligence can neither be satisfactorily diagnosed, nor explained by reference to a single test or a single mental process, yet the experimental evidence we have obtained suggests that a marked degree of the power to analyze a complex and abstract situation and to seize upon its implications is the most indispensable element in mathematical proficiency. The view advanced receives striking support in the high correlation between the composite for mathematical ability and the Interpolation test. The latter demands the ability to analyze abstract elements, to generalize from these and further to make application of the principle discovered. It is highly symptomatic of mathematical ability, and the factor common to it and to the other tests correlating closely with the composite is apparently this marked facility in the analysis of partial elements in a complex situation presented. The correspondences we have found suggest that mathematical intelligence embraces a wealth of functions, whose psychical nature is hard to detect and describe in detail, but if there is any community between these, it would seem to be of the kind described. Besides this general common factor, which is distributed in different amounts in all the tests that can be called mathematical, there are specific factors of varying importance. Thus in the Geometry test the common factor is present in considerable amount, but the specific differences between this test and the Interpolation test are obviously large. For a complete description of mathematical ability the latter are essential; but while these factors are necessary to success, the former still remains the most characteristic quality in mathematical mastery and tests possessing it

[32] Thorndike, E. L., Educational Psychology, II: Chap. 4.

in marked degree will serve best as an index to the presence of those qualities that determine success with the subject.

We have still, however, to evaluate both tests and composite by means of an *independent* estimate of the relative mathematical efficiency of the individuals tested. For this purpose the school marks in algebra and geometry up to date have been utilized. In the case of the Wadleigh High School group, two distinct sets of measurements were available, but this was lacking in the case of the Horace Mann group. Both the standard "Product-Moments" method and the Method of Ranks were applied to determine the extent to which the composite for mathematical ability was diagnostic of mathematical ability as measured by school records. The results can be most clearly presented in tabular form. Those obtained by the standard method are given first.

TABLE XXXI

PEARSON PRODUCT-MOMENTS METHOD

Group	Reliability Coefficients	Raw Coefficients	Average Crude Coefficients	Corrected Coefficients
Wadleigh High School	W_1 W_2=.88 Sch_1 Sch_2=.87	W_1 Sch_2=.59 W_2 Sch_1=.50	W Sch=.54	W Sch=.70
Horace Mann School	W_1 W_2=.85 Sch_1 Sch_2=.85 (assumed)	W_1 Sch_2=.75 W_2 Sch_1=.76	W Sch=.75	W Sch=.88

Here W_1 W_2 stand for the two applications of the tests included in the composite for mathematical ability and Sch_1 Sch_2 stand for the two independent series of school marks. In the case of the Horace Mann pupils, the sole available score (an average of four grades by the same teacher) was correlated with both applications of the composite of mathematical ability. In order to correct for attenuation, its reliability was assumed to be .85. It will certainly not differ greatly from .85 when calculated from the second series of records from Horace Mann, which will later be available.

The Method of Ranks gave the results tabulated below:

Group	Reliability Coefficients	Raw Coefficients	Average Raw Coefficients	Corrected Coefficients
Wadleigh High School	$W_1\ W_2=.89$ $Sch_1\ Sch_2=.80$	$W_1\ Sch_2=.70$ $W_2\ Sch_1=.53$	W Sch=.61	W Sch=.72
Horace Mann School	$W_1\ W_2=.95$ $Sch_1\ Sch_2=.85$ (assumed)	$W_1\ Sch_2=.79$ $W_2\ Sch_1=.79$	W Sch=.79	W Sch=.88

While the crude coefficients are of more significance than their theoretical corrections for practical diagnosis, the corrected coefficients give the most probable true amount of correspondence between the functions, which are represented by the two sets of measures. Considering the facts from Wadleigh High School, in which we have records for two years' work in the subject, we note that the composite foretells how well a pupil will do in his second year about three-fourths as accurately as does his entire record for the first year.

Using the corrected coefficients, we see that the degree of correspondence between the functions represented by the school marks and those represented by the composite is for the Wadleigh results, .70 and in the case of the Horace Mann results, .88. Combining these and giving the Horace Mann results double weight on account of their greater reliability, we find that the coefficient of correlation between school marks and the composite for mathematical ability is .82, a closeness of correspondence rarely found either in mental or physical measurements.

It is evident that our composite does measure the abilities fundamental to success in High School Mathematics, as measured by school records. The above coefficients indicate that in the composite we have a measure of the mental functions that are at work in the activities of the mathematics class. Accordingly our analysis of the characteristics that explain the correspondences observed between the traits measured by the tests is essentially an analysis of the qualities that determine successful accomplishment in the class room. Further, it has to be taken into consideration in judging of the kinship between the abilities covered by

the tests and the abilities functioning in school records that the two groups of subjects examined were unusually homogeneous in character as a result of an uncommonly careful and successful system of grading. Thus classification of pupils into groups of approximately equal ability was much more nearly attained in the case of these students than is at all usual. This tended to reduce the amount of correlation between school records and the tests and undoubtedly where a group of average variation is examined the aid the tests can lend towards classification will be yet greater.

Even in the case of these well graded schools, however, the difference in achievement of the various individuals tested was considerable. This indicates that the tests would prove of value in allocating pupils to a suitable group. In particular where our purpose is to select from a large number a class that can make rapid headway in this subject or at the other extreme a group whose likelihood of success is slight, the tests can afford invaluable aid in relegating each to his proper place. This would not only be of great benefit to the particular individuals concerned, but would put a difficult administrative task on a scientific footing. Thus the tests provide a means of measuring mathematical intelligence. By their aid we can determine in advance within known limits the relative standing of any individual in the subject. We can prophesy with a known degree of certainty from success with the tests a successful record in school work and contrariwise we can predict failure in the mathematics course from failure with them.

A further consideration which needs emphasis is that some of the tests owe their value to the very fact that they measure abilities which the school examinations fail to measure, but which are indispensable for success in future mathematical work. We would not expect perfect correspondence, therefore, between the composite for mathematical ability and the school records. Among such tests we would include the Matching Solids and Surfaces test, the Missing Steps in Series test, the Interpolation test and perhaps the Superposition test. These involve somewhat specialized traits. Such abilities as skill with spatial data, ability to think with space and to manipulate novel symbols, are important factors in future success. These are but poorly measured by

school examination marks and yet they are qualities crucial in the mastery of the subject. Inasmuch as the tests supply this lack, they offer valuable supplementation to the ordinary school-methods of measurement.

CHAPTER IV

THE PROGNOSIS OF MATHEMATICAL ABILITY

WE can thus bring the foregoing experimental results and theoretical conclusions to bear upon this last and most important of the problems with which this study is concerned: the practical prognosis of mathematical ability. The task in essence is to choose from the total list of tests a group, economical of time, easy of application, and possessing maximum diagnostic power. In making this selection our previous analysis of the factors that contribute towards success will lend invaluable aid.

Certain general principles serve to guide our choice in addition to the above practical considerations. Since the tests should cover as many phases of mathematical capacity as possible, those tests preëminently should be chosen which are both closely correlated with the composite for mathematical ability, and loosely correlated with each other. Guided by such experimental evidence of their nature as this study provides, together with existing knowledge of the characteristic qualities of those tests which have already been extensively used, we can provisionally select a set of tests whose value can later be established by the degree to which the results of their application coincide with a reliable independent estimate of the mathematical ability of the individuals examined.

Six tests were selected from the available seventeen to make this new Diagnostic Composite. These were the following: Algebraic Computation, Interpolation, Geometry Test, Superposition,[1] Mixed Relations, and the Trabue Language Scales.

Tables XXXII and XXXIII show the method by which the Diagnostic Composite was constructed.

[1] Modified Thurstone's Spacial Relations Test.

TABLE XXXII

WEIGHTS GIVEN TO THE TESTS INCLUDED IN THE DIAGNOSTIC COMPOSITE FOR MATHEMATICAL ABILITY

WADLEIGH HIGH SCHOOL

		Standard Deviation	*Average Standard Deviation*	*Desired Weight*	*Multiple*
Algebraic Computation	(1)	7.03	6.80	2	4
	(2)	6.57			
Interpolation	(1)	8.14	6.34	2	4
	(2)	4.55			
Geometry	(1)	5.25	5.12	2	5
	(2)	4.99			
Superposition	(1)	5.69	5.95	1	2
	(2)	6.22			
Mixed Relations	(1)	5.09	4.74	1	3
	(2)	4.39			
Trabue Language Scales	(1)	2.62	2.84	1	5
	(2)	3.07			

TABLE XXXIII

WEIGHTS GIVEN TO THE TESTS INCLUDED IN THE DIAGNOSTIC COMPOSITE FOR MATHEMATICAL ABILITY

HORACE MANN SCHOOL

		Standard Deviation	*Average Standard Deviation*	*Desired Weight*	*Multiple*
Algebraic Computation	(1)	6.29	6.78	2	6
	(2)	7.28			
Interpolation	(1)	37.77	39.48	2	1
	(2)	41.20			
Geometry	(1)	5.00	5.14	2	8
	(2)	5.29			
Superposition	(1)	5.82	6.60	1	3
	(2)	7.39			
Mixed Relations	(1)	9.62	9.16	1	2
	(2)	8.71			
Trabue Language Scales	(1)	3.20	3.64	1	6
	(2)	4.09			

It will be noted that these tests have been taken from each of the three main groups of abilities investigated, since our results have shown that each of these plays an important rôle in mathematical work. Further within these groups, tests correlating loosely with each other and closely with the composite for mathematical ability have been selected, wherever the subsidiary requirements of economy of time and ease of administration were fulfilled. The above sextet of tests can be applied in an hour and a half.[2] To prove the wisdom of our choice the resulting composite measures were correlated with the school marks obtained by each individual. The coefficients derived are summarized in the tables that follow:

PEARSON PRODUCT-MOMENTS METHOD

Group	Reliability Coefficients	Crude Coefficients	Average Raw Coefficients	Corrected Coefficients
Wadleigh High School	$\Delta_1\ \Delta_2 = .87$	$\Delta_1\ Sch_2 = .56$	$\Delta\ Sch = .62$	$\Delta\ Sch = .79$
	$Sch_1\ Sch_2 = .70$	$\Delta_2\ Sch_1 = .68$		
Horace Mann School	$\Delta_1\ \Delta_2 = .92$	$\Delta_1\ Sch = .94$	$\Delta\ Sch = .82$	$\Delta\ Sch = .92$
	$Sch_1\ Sch_2 = .85$ (assumed)	$\Delta_2\ Sch = .71$		

Δ_1 and Δ_2 represent the two applications of the tests included in the Diagnostic Composite for mathematical ability. Sch_1 and Sch_2 represent two independent series of school marks. As before, only one mark was available in the case of the Horace Mann girls.

METHOD OF RANKS

Group	Reliability Coefficients	Crude Coefficients	Average Crude Coefficients	Corrected Coefficients
Wadleigh High School	$\Delta_1\ \Delta_2 = .88$	$\Delta_1\ Sch_2 = .61$	$\Delta\ Sch = .55$	$\Delta\ Sch = .65$
	$Sch_1\ Sch_2 = .80$	$\Delta_2\ Sch_1 = .50$		
Horace Mann School	$\Delta_1\ \Delta_2 = .91$	$\Delta_1\ Sch_2 = .78$	$\Delta\ Sch = .77$	$\Delta\ Sch = .88$
	$Sch_1\ Sch_2 = .85$ (assumed)	$\Delta_2\ Sch_1 = .77$		

[2] A simple plan for the application and evaluation of these tests is given in the Appendix.

These figures offer substantial evidence of the significance of the six tests composing the Diagnostic Composite as measures of promise of mathematical performance in high school. The reliability of the composite is high and its prognostic power is such that in the case, for example, of the Wadleigh High School pupils one-half of the school record can be predicted with almost as great accuracy from the tests as from the other half. Thus an hour and a half spent on the tests may be expected to give a correlation of from .60 to .80 with future mathematical achievement. By means of these half dozen tests we are able to grade a group of pupils in an order of ability in mathematics and to classify them.

Further we are enabled to diagnose the lines of strength and of weakness in an individual's equipment for the subject, to discover, for instance, whether feeble intuitive grasp of spatial relations is the reason for failure with solid geometry, whether a poor command of language is the cause of lack of success with algebra problems. The tests are far from doing so with perfect precision. Investigation with many more tests upon a larger number of subjects would undoubtedly yield better methods of measurement. It is not only desirable to extend the tests and to supplement them, but to try others. Nevertheless, even in their present form they will prove useful, for by their aid we can predict with a known degree of accuracy the capacity of the pupil to undertake the high school course in mathematics. Not only because they measure abilities untested by ordinary examinations and important for success in the study of the subject, not only because they ascertain the ability of the pupil in greater detail, locating weaknesses or talent, but far more because they are exact measures, objective measures, which another can repeat and confirm or refute, they show themselves superior to the ordinary class examination and have a claim to consideration. They certainly will not have the same probable error and low reliability coefficients that characterize school marks. When we consider the results of Starch and Elliott's [3] investigation into the reliability of grading work in mathematics, the wide variation of grades given to the same paper by different teachers must arouse distrust

[3] Starch, D. and Elliott, E. C., Reliability of Grading Work in Mathematics, *School Review*, XXI: 254.

of conclusions founded upon such faulty data. Conclusions cannot be more trustworthy than the figures upon which they are based. The ordinary examination does not attempt to satisfy the conditions that the tests partially realize. The Diagnostic Composite can in an hour and a half provide a reasonably objective measure of the mathematical ability of the individual.

CHAPTER V

SUMMARIZED CONCLUSIONS

I. The crude coefficients of correlation between mathematical abilities for both groups combined range from .01 to .59. The corrected coefficients of correlation between mathematical abilities for both groups combined range from .06 to .82.

II. When mathematical ability is represented by a composite of all the mathematical tests, the highest correlation between that composite and any test is for the crude values .69 ± .05, and for the corrected values .81 ± .05.

III. The six best measures of mathematical ability, together with their correlations with the composite, are:

Crude	*r*	P.E.	*Corrected*	*r*	P.E.
Algebraic Computation ..	.69	.05	Algebraic Computation ..	.81	.05
Interpolation	.66	.04	Missing Steps in Series..	.78	.06
Missing Steps in Series..	.63	.06	Geometry	.76	.04
Geometry	.63	.05	Matching Equations and Problems	.76	.04
Superposition	.57	.02	Interpolation	.75	.02
Matching Equations and Problems	.57	.03	Arithmetic Problems	.74	.04

IV. No single test is a sufficient index to mathematical ability.

V. The functions represented by the three groups of tests for algebraic, geometrical, and verbal abilities are all equally essential in mathematical ability. The correlations between the composite for mathematical ability and each of these three groups of tests are practically the same.

VI. The correspondences found between the mathematical abilities tested may be traced to the common characteristic of capacity to react to partial elements in a situation. Mathematical ability is the complex resultant of a number of loosely connected capacities.

VII. Mathematical ability can be satisfactorily diagnosed by six tests requiring an hour and a half in time.

APPENDIX

TABLE XXXIV

ORIGINAL SCORES: WADLEIGH HIGH SCHOOL

	Algebraic Computation (1)	Algebraic Computation (2)	Matching Equations and Problems (1)	Matching Equations and Problems (2)	Matching Nth Terms and Series (1)	Matching Nth Terms and Series (2)	Interpolation (1)	Interpolation (2)	Missing Steps in Series (1)	Missing Steps in Series (2)	Inference with Symbols (1)	Inference with Symbols (2)	Geometry (1)	Geometry (2)	Superposition (1)	Superposition (2)	Symmetry (1)	Symmetry (2)	Matching Solids and Surfaces (1)	Matching Solids and Surfaces (2)
1	[illegible]	21	9	7	2	8	18	20	8	11	6	7	10	11	9	13	4	4	12	5
2	18	[illegible]	7	4	12	6	21	25	6	7	5	6	11	12	5	3	2	1	15	14
3	31	27	11	6	8	10	26	26	5	8	6	4	23	20	6	11	7	8	6	15
4	7	16	7	6	3	6	26	19	7	5	10	8	13	8	13	14	19	18	14	22
5	22	23	9	12	12	7	17	26	6	6	4	7	8	3	6	7	4	6	12	12
6	12	10	3	3	4	2	29	24	2	0	4	5	13	5	9	9	5	5	14	19
7	17	26	7	5	1	0	26	20	7	7	7	6	6	5	12	15	10	9	17	19
8	12	18	4	6	5	10	33	28	6	4	9	8	16	18	4	7	0	0	10	17
9	16	17	3	0	6	4	23	24	2	0	9	8	5	8	6	7	9	14	16	10
10	21	24	7	11	12	12	26	26	4	3	8	3	14	9	11	13	4	0	18	11
11	28	35	15	13	0	0	33	25	7	9	10	6	10	16	5	5	2	4	13	13
12	20	26	9	6	10	11	21	22	6	5	9	8	6	6	17	22	17	20	19	16
13	27	27	5	6	1	4	38	33	6	4	8	7	8	11	19	23	5	10	18	16
14	25	24	11	4	6	5	21	23	7	8	8	8	10	4	11	12	7	8	1	14
15	20	28	4	7	12	6	26	24	6	6	7	8	6	5	11	8	2	2	23	21
16	17	18	3	3	9	11	17	22	10	9	7	7	10	10	6	5	2	3	16	13
17	21	27	5	3	9	11	20	25	12	9	6	7	11	7	3	8	2	2	22	17
18	21	23	5	3	0	2	34	28	6	9	6	8	8	12	6	9	3	4	16	14
19	21	26	12	12	10	8	20	21	5	7	4	6	13	12	5	7	7	12	13	19
20	34	30	8	8	2	4	21	21	12	10	7	6	7	8	6	7	3	6	15	16

TABLE XXXIV—*Continued*

21	20	23	10	6	4	4	24	24	6	6	6	3	7	6	11	14	12	14	11	24
22	26	28	13	7	12	9	46	29	10	11	5	8	10	9	11	13	2	3	13	10
23	22	27	12	11	4	2	26	21	10	8	9	8	21	12	6	10	10	14	23	18
24	19	24	11	12	6	8	28	31	7	4	8	9	17	18	7	7	10	13	14	18
25	23	30	10	8	10	10	22	22	6	6	9	8	17	3	13	14	15	22	18	19
26	21	26	7	6	4	4	26	20	8	10	6	5	7	7	15	21	11	10	24	14
27	13	18	8	10	11	9	20	20	7	9	4	6	8	9	7	5	3	5	20	13
28	31	29	7	10	10	6	37	37	12	12	7	7	5	7	10	15	10	12	12	12
29	36	27	10	9	2	9	37	27	11	11	7	5	17	13	15	18	11	12	6	29
30	11	22	11	8	5	11	31	26	10	7	9	6	6	13	23	24	23	24	20	10
31	15	25	6	4	8	3	20	20	6	9	3	8	8	6	11	12	8	9	18	15
32	26	15	8	6	12	9	31	29	11	7	7	7	3	4	6	4	3	1	17	10
33	24	25	7	5	3	6	21	25	9	8	6	4	7	9	22	22	11	15	18	19
34	15	18	5	1	1	3	21	26	4	2	10	4	5	6	4	1	0	4	18	16
35	32	25	11	15	12	12	31	29	9	12	7	6	2	7	22	23	5	1	28	11
36	28	29	4	11	12	12	42	30	12	12	6	5	17	23	13	22	17	21	12	27
37	19	18	8	1	4	3	21	19	12	4	5	6	3	6	0	0	0	1	14	12
38	8	9	4	3	8	6	23	26	5	5	7	6	8	7	5	12	10	19	7	8
39	13	13	3	3	11	4	26	21	3	2	9	8	8	10	10	7	5	8	1	16
40	28	31	14	12	12	8	37	29	12	7	8	9	14	17	22	23	19	24	22	30
41	29	32	6	7	12	12	20	22	9	12	6	6	6	4	17	17	0	1	27	17
42	23	22	7	7	12	12	33	32	10	12	6	8	6	4	12	15	9	9	6	15
43	19	23	9	11	12	6	31	27	4	1	7	7	17	7	8	13	5	5	16	13
44	26	37	12	13	3	8	36	20	12	12	5	7	16	12	23	11	10	13	14	19
45	21	16	7	8	0	1	10	17	5	5	4	2	5	4	7	9	4	10	11	16
46	20	23	6	12	12	10	32	30	11	8	7	10	14	9	16	20	12	14	28	36
47	14	21	5	5	12	12	14	14	7	5	4	2	3	7	8	6	4	9	13	19
48	3	11	10	11	12	9	8	17	1	1	8	7	6	3	8	10	2	5	4	11
49	23	25	5	3	12	8	26	23	4	3	8	6	20	21	12	17	12	12	22	18
50	24	23	8	8	4	1	46	31	12	12	7	7	5	3	5	8	1	1	14	11
51	33	42	15	9	12	12	35	21	11	12	9	5	21	19	4	6	4	4	21	16
52	15	18	6	4	2	1	21	21	6	7	7	5	14	6	3	3	2	0	10	13
53	28	27	10	10	2	8	19	19	6	9	8	1	5	8	4	9	8	6	8	8
Av.	21.03 :	23.50	7.90 :	7.19	7.20 :	6.88	26.35 :	24.28	7.50 :	7.13	6.86 :	6.30	10.11 :	9.22	10.00 :	11.62	7.01 :	8.62	15.45 :	15.96
S.D.	7.03	6.57	3.13	3.55	4.33	3.62	8.14	4.55	3.03	3.39	1.73	1.74	5.25	4.99	5.69	6.22	5.43	6.55	6.19	5.65

TABLE XXXIV—*Continued*

ORIGINAL SCORES: WADLEIGH HIGH SCHOOL

	Geometrical Definitions (1)	Geometrical Definitions (2)	Reasoning (1)	Reasoning (2)	Arithmetic Problems (1)	Arithmetic Problems (2)	Mixed Relations (1)	Mixed Relations (2)	Logical Opposites (1)	Logical Opposites (2)	Trabue Language Scales (1)	Trabue Language Scales (2)	Thorndike Reading Tests (1)	Thorndike Reading Tests (2)	Age	School Marks (1)	School Marks (2)
1	13	22	9	11	1	2	11	12	67	81	6	2	17	20	200	40	66
2	17	16	13	10	3	1	5	7	57	78	6	2	11	18	181	44	74
3	15	[illegible]	5	10	1	1	6	13	78	90	6	12	16	31	179	59	68
4	20	24	8	10	3	2	17	19	78	90	10	10	22	24	194	37	25
5	21	24	12	12	1	0	7	11	76	72	6	6	16	24	179	71	55
6	17	26	10	13	2	0	8	15	67	75	7	9	13	28	174	38	53
7	15	13	11	9	2	0	16	18	73	81	3	4	18	20	172	56	55
8	23	9	6	9	1	1	16	17	80	84	8	7	15	26	174	44	58
9	22	23	8	10	3	2	12	12	66	81	10	6	14	23	189	45	69
10	9	15	5	8	2	1	2	0	66	72	7	2	20	26	176	47	77
11	18	15	14	16	3	2	16	18	75	87	12	12	15	24	183	63	55
12	19	27	13	18	4	0	15	17	74	87	10	10	22	36	199	83	55
13	21	20	19	18	3	3	16	18	74	87	2	4	17	24	170	51	80
14	18	19	8	10	1	3	11	17	71	84	8	9	17	26	159	58	60
15	19	20	6	11	2	0	16	19	85	87	10	5	18	23	170	52	91
16	13	14	12	13	2	1	16	13	83	87	7	5	11	17	172	50	75
17	18	18	5	6	3	2	4	18	78	81	9	6	19	23	168	56	80
18	24	15	13	13	3	0	15	17	87	87	10	14	17	20	183	61	75

TABLE XXXIV—*Continued*

19	25	26	10	6	2	2	5	16	74	81	10	10	20	30	170	74	78
20	17	17	4	13	2	1	17	17	67	87	5	9	15	21	171	61	63
21	16	20	14	9	3	1	17	17	73	81	6	7	19	19	166	58	71
22	20	14	9	12	4	3	10	15	80	81	11	13	15	22	172	44	85
23	19	18	12	6	2	3	17	20	73	84	9	7	17	24	176	76	55
24	14	13	17	19	5	5	6	19	76	90	13	10	19	30	150	69	83
25	28	27	4	12	1	1	4	14	82	84	10	13	20	29	173	67	80
26	14	20	5	5	3	2	9	16	69	82	6	7	15	21	169	71	63
27	18	17	7	10	2	0	8	18	78	87	2	6	19	22	178	63	50
28	24	18	13	15	5	3	4	6	74	84	8	12	19	26	165	70	66
29	32	24	16	12	4	1	6	17	79	84	10	9	15	25	158	60	69
30	22	16	11	13	2	2	13	13	69	87	17	6	12	22	186	45	72
31	13	13	7	9	4	2	1	9	64	72	8	4	13	18	168	42	69
32	9	11	8	11	4	1	4	15	73	84	7	6	21	25	183	50	55
33	15	15	13	10	3	0	16	11	69	63	7	2	12	19	174	44	70
34	7	14	6	11	0	0	5	9	48	66	6	5	16	12	177	58	68
35	13	13	8	16	3	1	13	17	81	81	12	9	17	22	176	67	50
36	20	23	9	14	5	2	17	15	84	87	11	8	18	22	180	66	70
37	17	21	11	8	3	0	7	4	81	81	11	5	14	24	177	43	74
38	12	11	12	7	5	3	8	6	50	66	6	4	15	20	177	50	63
39	22	16	9	7	3	1	8	7	75	84	7	4	14	19	162	42	63
40	24	21	18	10	5	2	5	20	79	84	11	14	20	24	168	63	65
41	14	13	14	15	1	1	1	13	71	81	9	8	11	17	181	70	55
42	19	21	11	14	3	3	7	11	83	84	6	11	14	23	155	50	80
43	19	19	4	14	3	1	10	18	72	81	8	5	18	23	164	61	63
44	21	21	13	19	3	4	4	18	77	84	8	6	12	18	167	75	72
45	11	15	7	11	5	2	3	10	67	81	5	6	17	16	194	73	50
46	21	23	19	17	5	5	11	17	75	87	10	8	17	24	172	73	73
47	13	15	9	13	0	2	11	10	68	72	9	4	13	20	187	41	60
48	0	24	9	13	0	0	16	15	72	84	8	8	18	22	173	50	74
49	29	24	5	13	1	0	6	19	83	81	12	7	17	23	178	61	60
50	18	13	18	13	3	1	17	13	77	87	6	6	17	23	163	56	70
51	14	23	12	17	2	3	17	17	73	90	13	8	13	25	164	81	94
52	21	16	8	13	4	3	10	14	73	81	13	8	17	31	179	67	60
53	16	13	12	11	3	3	12	13	84	90	11	8	12	20	182	52	75
Av.	17.62 :	18.45	10.20 :	11.77	2.69 :	1.60	10.07 :	14.15	73.73 :	82.15	8.26 :	7.32	16.26 :	22.90	174.66	57.64 :	66.77
S.D.	5.58	4.85	3.89	3.37	1.36	1.27	5.09	4.39	7.78	6.17	2.62	3.07	2.91	4.22	3.49	12.09	14.58

TABLE XXXV

ORIGINAL SCORES: HORACE MANN SCHOOL

	Algebraic Computation (1)	Algebraic Computation (2)	Matching Equation and Problems (1)	Matching Equations and Problem (2)	Matching Nth Terms and Series (1)	Matching Nth Terms and Series (2)	Interpolation (1)	Interpolation (2)	Missing Steps in Series (1)	Missing Steps in Series (2)	Inference with Symbols (1)	Inference with Symbols (2)	Geometry (1)	Geometry (2)	Superposition (1)	Superposition (2)	Symmetry (1)	Symmetry (2)	Matching Solids and Surfaces (1)	Matching Solids and Surfaces (2)
101	[illegible]	27	6	9	12	2	94	109	7	2	25	28	17	13	4	2	7	7	31	23
102	[illegible]	17	7	7	11	5	93	79	3	5	14	15	12	11	18	22	27	25	19	29
103	26	[illegible]	11	16	2	2	86	73	5	3	25	27	11	10	16	26	10	15	16	13
104	15	18	6	15	12	15	53	79	2	3	39	53	13	17	10	12	12	15	8	23
105	18	11	7	23	6	1	105	81	6	6	45	38	9	7	9	13	18	19	10	11
106	1	1	3	10	4	4	32	31	2	1	3	9	4	2	4	1	3	5	16	7
107	16	18	11	24	10	11	73	79	6	5	33	36	9	13	16	7	11	14	29	22
108	11	18	8	10	2	1	75	94	4	4	34	28	16	19	15	17	24	24	23	32
109	14	17	16	14	10	8	84	99	6	4	31	44	16	10	20	21	29	34	36	39
110	16	14	5	7	0	0	58	43	4	2	37	43	15	16	8	8	4	4	25	30
111	26	26	[illegible]	33	8	9	96	110	7	3	49	64	4	9	11	13	4	5	19	20
112	27	28	15	20	18	19	142	126	7	5	40	45	13	14	16	13	17	23	37	37
113	18	19	8	13	11	7	110	144	6	5	34	51	13	8	14	19	19	25	29	33
114	15	9	8	13	10	8	40	33	1	1	21	41	3	5	7	3	13	10	7	17
115	19	14	11	[illegible]	22	23	72	62	6	2	39	37	10	8	6	5	2	1	18	14
116	22	24	11	23	13	10	107	101	3	6	32	42	17	16	13	28	32	35	27	23
117	29	21	23	28	12	12	74	51	6	6	45	38	15	20	19	21	21	28	17	9
118	34	31	20	34	11	13	174	164	9	11	40	46	23	25	19	14	20	22	31	21
119	22	20	7	21	11	15	133	166	9	6	39	42	15	12	10	12	12	17	18	24
120	26	24	10	16	16	14	162	158	7	7	56	50	21	20	24	20	20	23	33	28

TABLE XXXV—*Continued*

121	17	17	8	19	10	7	65	79	3	3	37	34	21	13	9	7	10	8	18	18
122	18	20	10	16	3	3	97	102	6	3	41	54	12	11	12	16	21	20	28	22
123	25	30	17	27	12	14	139	190	7	8	29	40	21	17	19	14	15	19	32	33
124	31	22	11	22	16	15	83	118	6	6	39	47	15	11	6	14	4	7	18	18
125	24	22	8	31	10	13	109	98	5	4	55	52	18	18	14	19	12	4	38	27
126	25	18	8	12	11	10	133	124	5	4	35	45	7	3	3	1	11	8	13	7
127	26	24	11	25	12	15	75	108	4	6	45	42	6	9	7	6	15	15	18	16
128	19	21	13	16	14	7	91	78	4	2	40	60	4	5	19	22	13	16	22	16
129	22	30	11	22	8	2	186	142	6	8	31	39	9	10	28	35	33	36	27	25
130	28	29	11	21	15	6	95	102	6	5	22	29	7	10	10	10	0	0	7	17
131	12	24	1	12	1	2	15	13	0	0	26	30	13	7	20	20	1	0	31	31
132	18	23	16	13	12	13	30	25	2	3	24	19	11	9	15	15	19	25	19	14
133	23	17	13	30	8	4	76	79	5	3	30	42	9	3	4	4	2	2	17	22
134	17	24	11	18	14	13	112	129	6	7	40	40	15	15	12	21	18	15	30	20
135	26	27	8	23	23	23	132	118	6	3	31	32	16	16	11	15	8	15	25	32
136	26	31	18	20	10	16	154	98	6	6	42	47	15	18	10	9	1	3	21	16
137	16	15	13	18	3	1	93	86	4	3	26	25	7	7	6	4	1	0	17	14
138	17	18	15	14	5	8	70	76	4	2	26	26	12	11	11	13	13	14	21	16
139	21	14	4	17	17	9	44	43	2	2	33	39	11	6	10	12	6	11	23	23
140	11	8	5	10	6	2	37	32	3	2	16	35	12	1	11	7	0	0	16	10
141	14	9	10	14	9	8	79	78	1	2	46	38	10	8	14	11	1	2	8	5
142	16	13	6	19	12	10	72	79	6	6	31	29	10	1	3	6	13	12	17	9
143	31	31	9	23	19	22	116	117	6	7	34	41	14	13	14	10	24	26	13	14
144	25	13	5	12	10	7	48	50	5	5	56	50	11	4	4	5	1	0	29	26
145	8	7	4	5	12	10	135	123	3	2	19	26	8	7	6	6	0	0	20	16
146	19	21	9	16	11	8	66	52	4	3	13	24	6	0	7	2	6	8	4	12
147	15	17	7	5	10	8	51	70	6	8	19	24	4	9	0	1	4	8	22	19
148	27	24	13	26	4	0	139	117	6	2	33	53	19	16	13	16	18	18	31	36
149	31	37	25	32	20	22	133	108	8	8	47	51	24	16	22	26	24	30	16	31
150	16	8	3	8	0	4	74	89	3	2	21	35	4	4	13	13	14	20	13	21
151	18	17	6	9	15	15	53	48	5	4	15	23	6	8	12	13	22	22	23	19
152	29	34	12	29	10	17	187	214	10	6	56	68	15	16	17	22	0	0	11	18
153	10	14	12	12	21	15	109	133	5	6	25	31	12	12	10	7	6	11	19	9
154	19	21	10	20	8	1	109	108	3	4	27	29	15	11	11	10	7	8	34	27
155	11	11	7	16	8	3	28	31	1	1	12	32	7	11	13	7	6	9	19	17
156	15	17	8	13	13	8	93	94	6	4	41	42	8	4	13	12	15	17	26	23
157	29	25	20	25	15	13	77	85	7	6	33	36	16	14	22	17	6	7	14	15
158	28	29	18	17	17	20	126	140	8	7	44	54	12	12	16	12	8	10	21	20
159	30	28	19	30	7	9	113	116	8	10	42	51	15	13	18	20	17	17	37	29
160	16	15	10	7	6	1	25	26	2	1	22	26	10	6	6	5	4	3	23	18
161	15	20	8	22	5	2	79	71	3	3	34	37	3	9	4	6	2	5	17	10
Av.	20.42 :	20.11	10.39 :	18.19	10.54 :	9.26	91.98 :	92.96	4.95 :	4.32	33.09 :	38.91	11.96 :	10.63	12.03 :	12.59	11.57 :	13.14	21.42 :	20.42
S.D.	6.69	7.28	4.95	7.67	5.28	6.26	37.77	41.20	2.15	2.34	11.48	11.62	5.00	5.29	5.82	7.39	8.69	9.68	8.06	8.03

TABLE XXXV—*Continued*

ORIGINAL SCORES: HORACE MANN SCHOOL

	Geometrical Definitions (1)	Geometrical Definitions (2)	Reasoning (1)	Reasoning (2)	Arithmetic Problems (1)	Arithmetic Problems (2)	Mixed Relations (1)	Mixed Relations (2)	Logical Opposites (1)	Logical Opposites (2)	Trabue Language Scales (1)	Trabue Language Scales (2)	Thorndike Reading Tests (1)	Thorndike Reading Tests (2)	Age	School Marks
101	24	21	12	14	2	1	23	29	125	133	13	13	15	83	174	12
102	27	22	7	9	2	1	22	20	125	130	22	17	11	82	186	13
103	20	20	10	8	3	2	25	29	113	113	14	6	18	78	174	12
104	25	30	14	9	3	2	7	22	88	81	17	15	19	91	165	12
105	21	19	6	12	2	2	33	27	126	149	10	16	16	69	188	11
106	10	16	4	9	2	1	18	23	76	110	10	10	15	56	198	5
107	12	16	12	15	3	2	26	29	138	138	20	11	21	89	179	12
108	18	24	13	11	2	0	25	25	119	139	19	12	16	77	173	13
109	27	27	19	19	3	3	29	28	129	151	19	16	22	90	176	10.5
110	25	24	15	13	1	1	23	22	126	106	18	14	20	88	168	11
111	21	18	7	7	3	4	21	19	124	113	11	11	15	68	177	16
112	27	25	17	18	5	4	21	19	140	152	18	19	23	88	184	12
113	21	19	6	6	4	2	27	26	141	148	16	17	19	71	169	12
114	11	11	7	6	2	1	13	16	88	123	17	10	16	70	203	11
115	19	19	14	13	3	1	9	2	143	132	18	19	17	84	179	12
116	22	19	11	13	3	4	24	28	142	141	20	22	19	84	157	16
117	26	22	18	18	4	1	32	32	146	160	19	14	22	87	168	16
118	35	31	12	15	5	4	34	39	159	198	22	26	23	92	155	13.5
119	20	24	6	14	5	4	24	18	128	157	19	19	21	89	169	14
120	29	26	13	15	5	3	22	25	135	130	16	19	19	79	167	15
121	20	24	8	9	2	2	21	22	125	137	15	15	18	88	164	14
122	28	24	16	14	4	3	23	27	126	136	16	17	22	86	180	14

TABLE XXXV—*Continued*

123	25	23	16	14	4	5	30	34	158	159	21	24	19	91	164	18
124	20	22	13	13	4	2	35	37	104	176	18	19	22	88	172	14
125	28	23	18	16	5	0	27	32	135	144	12	21	18	74	178	16
126	9	5	8	11	3	2	23	31	123	142	21	17	21	76	154	13
127	23	25	11	12	2	2	32	35	171	193	14	16	21	84	165	15
128	15	14	9	10	1	0	7	10	130	104	14	11	16	69	179	13
129	26	21	19	18	5	4	32	24	139	166	18	18	21	90	188	15
130	20	21	9	8	3	1	7	31	111	147	15	21	22	87	166	13
131	24	25	16	17	4	3	27	27	141	174	14	20	18	78	175	11
132	11	10	10	10	3	3	25	27	151	153	22	19	19	84	196	11
133	14	14	9	13	3	2	4	10	153	182	13	15	21	75	181	11
134	18	26	12	13	3	3	31	23	132	183	18	17	17	85	164	13
135	27	21	15	16	3	2	19	27	148	138	17	16	19	87	187	16
136	20	19	9	13	3	1	31	31	136	168	14	17	22	82	178	17
137	14	19	5	7	2	1	19	18	149	128	15	12	18	69	177	7
138	25	26	10	14	3	2	21	22	95	114	19	21	20	77	176	9
139	16	10	9	11	3	2	3	24	148	140	20	14	16	71	198	11
140	12	8	8	8	1	1	4	6	109	150	13	12	12	55	195	7
141	12	11	12	13	0	1	5	0	124	171	17	22	18	84	156	8
142	21	20	8	10	4	3	17	18	108	129	19	22	24	87	170	12
143	17	14	15	14	3	4	3	1	126	131	13	10	15	83	159	18
144	27	27	14	19	4	2	28	30	136	135	18	20	19	79	163	13
145	11	14	6	12	1	2	24	28	117	135	17	17	16	76	174	7
146	16	7	6	9	5	4	18	21	125	128	17	17	17	79	185	13
147	22	21	12	13	3	2	28	28	185	177	22	17	19	85	189	12
148	27	26	11	15	5	3	29	24	127	160	18	18	21	85	162	16
149	31	28	10	13	5	5	37	39	150	156	20	21	20	90	167	17
150	17	21	7	9	1	3	5	14	124	141	10	14	20	71	182	6
151	9	11	9	12	2	1	15	23	83	124	19	16	14	73	176	8
152	16	20	12	15	5	6	24	26	160	152	14	17	18	82	165	14
153	25	24	9	9	1	2	23	23	122	156	14	12	16	75	178	10
154	22	25	7	9	2	1	8	22	127	147	15	10	19	79	171	15
155	25	21	12	12	2	2	20	23	117	115	15	22	20	86	199	7
156	21	22	15	16	4	3	21	17	116	134	14	20	17	87	166	12
157	31	29	6	15	3	2	1	12	156	170	11	22	17	81	163	15
158	24	23	12	10	6	3	32	34	120	144	13	18	18	88	165	14
159	23	25	13	12	6	4	27	27	139	146	15	24	20	89	167	16
160	13	11	10	13	1	0	5	6	127	127	14	16	18	68	188	8
161	18	17	5	9	3	0	7	18	119	142	20	20	20	84	180	11
Av.	20.60 :	20.16	10.88 :	12.29	3.15 :	2.24	20.59 :	23.11	129.55 :	143.57	16.42 :	16.77	18.50 :	80.68	174.93	12.44
S.D.	6.03	5.91	3.78	3.22	1.37	1.35	9.62	8.71	19.39	23.02	3.20	4.09	2.77	8.22	11.60	3.02

INSTRUCTIONS TO SUBJECTS

The directions used in the case of the mathematical tests devised for the first time are described in great detail in order that it may be possible for any one to repeat their application from the given description with sufficient similarity of procedure to permit a comparison of the results obtained with those recorded in this study.

The instructions given to the two groups examined were identical. The time limit, however, was different in the case of several tests, which had been extended, before application to the second group.

A regular procedure in general technique was followed for all the tests. Thus they were invariably presented face down with the warning: "Do not turn your paper until you are given the signal 'Go' and stop at once, when you hear the signal 'Stop.'" This was followed by the command: "Write your name and the date in the upper right-hand corner."

ALGEBRAIC COMPUTATION:

"On the other side of the paper in front of you are problems in algebra. Work them in the order given. First do 1, then do 2, then 3, and so on."

Time limit: Wadleigh High School, 7 minutes
Horace Mann School, 12 minutes
(Sheet I, 4 minutes)
(Sheet Ia, 8 minutes)

MATCHING EQUATIONS AND PROBLEMS:

"Read the directions. On the other side of the sheet in front of you are 12 problems and 12 equations which stand for them. Each problem corresponds to one equation, and only one, and each equation stands for one problem and only one. You have to match the problems and equations. Do not find the answers to the problems. Do not solve the equations. Only match the equations and problems.

"First read problem 1 and state it in the form of an equation, then look down the list of equations till you find the right one corresponding to problem 1. Then write 1 opposite the equation."

Time limit: Wadleigh High School (1), 4 minutes
(1-a), 3 minutes
Horace Mann School (1), 4 minutes
(1-a), 10 minutes

MATCHING NTH TERMS AND SERIES:

"Read the directions."

Write $2n$ on the board. Ask, "If we substitute for n, 1, what does $2n$ equal? If we substitute 2 for n, what does $2n$ equal? (and so on), 2, 4, 6, 8, 10...... That is a series. How is it formed? How is 4 got from 2, and 6 from 4, and 8 from 6? What would be the next number after 10?

2, 4, 6, 8, 10, 12, 14, and so on. This series corresponds, or is derived from the formula, $2n$. The formula is a short way of writing the series 2, 4, 6, 8...... Take the formula $5n$-1, what is the series derived from it? First let n equal 1, then the formula equals 4, let n equal 2, then the formula equals 9, next 14, next 19, next 24, next 29. This series corresponds to $5n$-1. What is the term following 29? How is the series found?

"5, 9, 14, 19, 24......corresponds to $5n$-1.

"$5n$-1 is a short way of writing the series 4, 9, 14, 19, 24......

"On the opposite side of this page are 12 formulae and 12 series derived from them. You are to pair these correctly, writing in Column 3 opposite each formula, the number of the series obtained from it. Thus, suppose the series obtained from the first formula were the 7th, then you would write in Column 3, opposite the first formula the number 7. First look at formula I, get from it the series for which it stands. Look for the series among the 12 given series and write the number of the one selected in Column 3."

Time limit: Wadleigh High School, 2 minutes.
Horace Mann School (1), 1½ minutes.
(1a), 2½ minutes.

INTERPOLATION TEST:

Write on the blackboard 2, 4, 6, 8, 10, 12......
Ask: "What is the rule for making this series?
How is each term got from the one before it?
How is 4 got from 2? 6 from 4, 8 from 6, etc?
2, 4, —, 8, 10, —, 14......
What are the missing numbers?
5, 10, —, 20, 25, —, —, 40......
What are the missing steps in this series?
1, 4, —, 10, —, —, —, 22......
What are the missing steps in this series?"

"Write your name and the date. Lay down your pencil."

"On the other side of this sheet are similar series. They increase in difficulty. In the first there are only 2 steps missing, but more and more steps are missing, as you go on. You have to fill up each blank space with one missing number."

"Do not turn your paper till I say 'Go' and stop immediately when I say 'Stop,' laying down your pencil."

"You will have — minutes. Your score depends on the number of blanks correctly filled."

Time limit: Wadleigh High School, 2 minutes.
Horace Mann School (1), 8 minutes.
(1a), 5 minutes.

MISSING STEPS IN SERIES:

"Read the directions."

"Last day you had to find missing numbers in series that were got by additions. Always you had to find the number that was added and then you were able to fill in the blanks. This time the series are made not only by addition, but also by subtraction, multiplication, and division. You have to find out in each case which it is. Discover the rule and so supply the missing numbers. Look at the illustrations. What is the rule for the first? What is the missing number? What is the rule for the second? What is the missing number? the third? the fourth?"

"Do not turn your paper till I say 'Go,' then turn at once and as fast as you can write in the missing numbers. When I say 'Stop,' at once lay down your pencil. You will have one minute. Your score depends on the number of blanks correctly filled.

Time limit: Wadleigh High School, 1 minute.
Horace Mann School, 1 minute.

INFERENCE WITH SYMBOLS:

"Write your name and the date." Read the directions.

"Do not turn over the sheet of paper until told."

"In algebra you work with symbols. You have already learned to use plus (+) for add, and minus (—) for subtract, and equals (=) for equals. Now on this sheet you have new symbols. Look at the illustrations. The first reads "*A* is greater than *B*, *B* equals *C*, therefore, —? *A* is greater than *C*. The second reads *A* is greater than *B*, *B* is not less than *C*, therefore, —? *A* is greater than *C*."

On the other side there are similar inferences. You are to find the conclusions from the "Given Facts" by "filling in" the correct symbols. That is, you make an inference from given statements, e.g., if I say *A* is the brother of *B*, and *B* is the brother of *C*, what is the relation between *A* and *C*, you can tell me that *A* is the brother of *C*. The first problems are easy: they become more difficult towards the end. Note, where none of the symbols give a true conclusion, draw a line. There are cases where it is impossible to find any conclusion, i.e., where you cannot say whether *A* is greater than *B*, less than *B*, equal to *B*, not greater than *B*, or not less than *B*.

Time limit: Wadleigh High School, 6 minutes.
Horace Mann School (1 and 1a), 15 minutes.
(2 and 2a), 15 minutes.

GEOMETRY TEST:

Hand out the reference and problem sheets. Say: "Read the reference sheet." Then go over directions carefully with the group. Explain the illustration. After they have tried Problem 1, explain it.

N. B. Say: "You may require more than one of the facts to solve the

later problems. Be sure and give them all. You will get a mark for each correct reference."

"Write your name and the date on the back of the Problem sheet."

(Answer questions)

Time limit: 30 minutes.

SUPERPOSITION TEST:

Prepare three cardboards similar to the three cards shown on the instructions side of the Spatial Relations test. Make these cards about 10 inches on the side. Paint one edge of the card black on both sides of the card. Cut the holes as indicated.

Before giving the test draw on the blackboard the complete drawing on the instructions side of the blank. This need not be very accurately done. Warn the group that the instructions must be attended to very carefully to be understood. Repeat orally the following, while moving one of the large cardboards into place on the blackboard drawing.

"Suppose that the figure with a circle in it is a small card with one of its edges painted black and with a hole in one corner.

"If this card is moved around so that its black edge lies upon the long heavy black line, it will fit one of those two figures shown.

"Decide which it fits and then with your pencil draw a circle where the hole would be."

Give this paragraph verbatim for each of the three tests on the instructions side and also make the group try these three tests.

Then say: "Do the same thing with the other outlines given as quickly as you can."

Time limit: 1 minute.

This test was applied twice in the case of the Wadleigh High School pupils and four times in the case of the Horace Mann group, applications being made on two different days. In order to obtain two measures of the ability tested the scores for the first two applications were added and similarly for the last two applications.

SYMMETRY TEST (THURSTONE SPATIAL RELATIONS TEST):

Prepare three cardboards similar to the three cards shown on the instructions side of the sheet. Make these cards about ten inches on the side. Paint one edge of each card black on both sides of the card. Cut the holes as indicated.

Before giving the test draw on the blackboard the complete drawing on the instructions side of the blank. This need not be very accurately done.

Allow two minutes for the group to read the instructions, warning them that the instructions must be read very carefully to be understood.

At the end of this time, while moving one of the large cardboards into place on the blackboard, repeat orally this paragraph verbatim for each of the three cards on the instruction side.

"Imagine that this card is picked up, turned over and placed face down with the black edge of the card touching the long heavy black line to the right. Imagine the card moved along this black line until its edges fit the edges of one or the other of the lozenge shaped outlines.

"With your pencil, draw a circle in the corner where the hole will be."

Time limit: Wadleigh High School, 2 minutes.
Horace Mann School, 2 minutes.

This test was applied twice to the Wadleigh High School group and three times to the Horace Mann pupils. It had been intended to give four applications in all to the latter; but time was not available. The usual plan was followed of obtaining two measures of the ability tested. The sum of the alternate scores in the third application was added to the sum of the scores in the first application and the second application respectively, so giving two comparable measures of ability in applying the principle of symmetry.

MATCHING SOLIDS AND SURFACES:

Give out the reference and the test sheets.

Allow five minutes for reading the former. Show the actual solids. Go over the directions with care, explaining in detail: (1) Matching solids and surfaces, (2) method of cutting solids and different surfaces obtained by vertical, horizontal, slanting cuts, (3) method of lettering, answering any questions upon this.

Time limit: 30 minutes.

GEOMETRICAL DEFINITIONS TEST:

"Read the reference sheet. Do it carefully."
(Allow two minutes for this)

"Write your name and the date at the right hand top corner of the second sheet."

"On the reference sheet there are drawings of geometrical figures, and definitions of these figures. On the second sheet are different figures and you are asked to give complete definitions of these. The reference sheet shows the kind of definitions that is wanted. Notice you must give a complete and correct definition. The definitions of the new figures will be similar, but not exactly the same."

Time limit 15 minutes; usually 90% finish at 12 minutes.

REASONING TEST:

Read over the directions with the class and show how the illustrative examples are worked. Also say, "The first arguments are very simple but grow more and more difficult. You have at first to make only one inference, but towards the end you have to make two or three consecutive inferences to get the answer and find the required relation."

Time limit: Wadleigh High School, 10 minutes.
Horace Mann School, 10½ minutes.

ARITHMETIC PROBLEMS:

"Do the problems in the order given, first 1, then 2, then 3, and so on."

Time limit: Wadleigh High School, 10 minutes.
Horace Mann School, 10 minutes.

MIXED RELATIONS TEST:

Write on the board: color—red, name—John.
page—book handle—
fire—burns soldiers—

"The first pair of words express a certain relation.

"You have to find a fourth word, which along with the third word will give the same relation.

Thus color—red, name—John. That is, Red is a color and John is a name.

"Then ask: Page is to book as handle is to what? That is: Page is a part of a book and handle or blade is part of a knife.

"Then ask: fire burns soldiers—what?

"On the other side of the sheet similar relations have to be found. You must find a fourth word that is related to the third, as the second is related to the first.

"Your score depends on the number of correct answers."

Time limit: Wadleigh High School, 1½ minutes.
Horace Mann School, 3 minutes.

LOGICAL OPPOSITES TEST:

"What is the opposite of better?"
"What is the opposite of friend?"
"What is the opposite of true?"
Point out that the answer "*untrue*" is not as good as "*false*."

"On the other side of this sheet there is a list of words. You are to write after each one a word that is opposite in meaning to it. You will have a minute and a half. Your score depends on the number of right opposites written."

"If you cannot think of the correct opposite within ten seconds, go on to the next word."

Time limit: Wadleigh High School, 1½ minutes.
Horace Mann School, 5 minutes.

TRABUE LANGUAGE SCALES (L, M, J, K):

Standard directions were followed.[1]

[1] See Trabue, M. R., Completion-Test Language Scales, Teachers College, Columbia University, Contributions to Education, No. 77.

THORNDIKE READING TESTS [2]

Wadleigh High School:

(1) Scale Alpha 2

(2) Tests L, M, N, X, B, W

Horace Mann School:

(1) Scale Alpha 2

(2) Tests L, M, N, X, R

"Do exactly what it asks you to do. Answer every question. Bring your paper when you have finished so as to get credit for quick work, but work very carefully.

Time limit: 30 minutes.

[2] See *Teachers College Record*, September, 1914, November, 1915, and January, 1916.

THE PRACTICAL USE OF THE SEXTET OF TESTS FOR DIAGNOSING MATHEMATICAL ABILITY

The object of this sextet of tests is to provide a quick means of diagnosing the mathematical intelligence of pupils in the third year of the Junior High School [3] with a view to improving the classification of students in high school by eliminating from the mathematics classes those unfit for further mathematical training and selecting those capable of progressing at a more rapid rate than the majority. The tests also serve to discover particular lines of mathematical weakness.

The tests recommended are Algebraic Computation, Interpolation, Geometry, Superposition,[4] Mixed Relations, and the Trabue Language Scales L and J.[5] They are designed to measure the more important phases of mathematical capacity demanded by high school mathematics and in particular the ability to manipulate numerical and algebraic symbols, the ability to grasp and handle spatial relations, and the ability to deal effectively with words. They are of such a nature as to enable an intelligent teacher to form an independent estimate of the pupil's mathematical capacity and likelihood of success in future mathematical work. They measure original ability rather than the effects of training.

The tests have been applied under differing conditions, however, to several hundred persons. The results presented here as most valuable for purposes of comparison are those obtained from sixty-one pupils in the third year of the Junior High School of

[3] The tests can be given in the seventh and eighth grades. The time limits must in these cases be considerably extended and comparative standards have not been tabulated.

[4] This is a modified form of the Thurstone Spatial Relations test.

[5] See pages 17 to 41 for a description of these tests. The blanks for the Superposition test can be obtained from L. L. Thurstone, Carnegie Institute of Technology, Pittsburg. The Trabue Language Scales can be procured from the Bureau of Publications, Teachers College, Columbia University.

the Horace Mann School for Girls. The tests should be administered under conditions precisely similar to those present in their case. They should therefore be given during the second half of the school year and the same method of scoring should be followed.[6]

The application of the tests demands 72 minutes and if we allow for the preliminary explanations which are necessary, at least an hour and a half in time is required for obtaining comparable results. The Trabue Language Scales and the Mixed Relations test can conveniently be given in the English class hour as a class exercise. Two mathematical periods will then complete the application of the four remaining tests.

The following arrangement is suggested as satisfactory.

Class Period (40 minutes)	Name of Test	Time for Preliminary Explanation	Time for Test
I	Superposition	5 minutes	2 minutes (1 minute for each application)
	Algebraic Computation		12 minutes (4 minutes for sheet 1 and 8 minutes for sheet 1a)
	Interpolation	5 minutes	13 minutes (8 minutes for sheet 1 and 5 minutes for sheet 1a)
II	Geometry	10 minutes	30 minutes
III	Mixed Relations	5 minutes	3 minutes
	Trabue Language Completion Scales	2 minutes	10 minutes

The results so obtained should be treated in the following way. Each individual's score in each test should be first expressed as a deviation from the average mark obtained by the Horace Mann group.[7]

[6] For instructions to subjects see the Appendix and for method of scoring see the tests, pages 17-41.

[7] The individual scores might also be expressed as deviations from their own class average.

The Horace Mann averages were as follows:[8]

Algebraic Computation	20
Interpolation	92
Geometry	12
Superposition	12
Mixed Relations	21
Trabue Scales L and J	16

The general principle underlying the estimation of mathematical intelligence is that as many phases as possible of mathematical skill and insight should be tested and the results pooled. In order to accomplish this it is essential to make the variabilities of the various tests equal.[9] This is done by multiplying the deviations for the Algebraic Computation test by 6, for the Interpolation test by 1, for the Geometry test by 8, for the Superposition test by 3, for the Mixed Relations test by 2, and for the Trabue Scales by 6. These new deviations should then be summed algebraically for each individual. The resulting number gives a measure of his mathematical capacity.

In the case of the Horace Mann pupils the composites scores so obtained were as follows:

Individual	*Score*	*Individual*	*Score*
1	34	31	—93
2	69	32	—29
3	30	33	—98
4	—89	34	50
5	—44	35	107
6	—304	36	124
7	—21	37	—91
8	—4	38	—25
9	46	39	—68
10	—30	40	—164
11	—51	41	—85
12	124	42	—77
13	32	43	58

[8] The actual averages obtained by the Horace Mann group were:

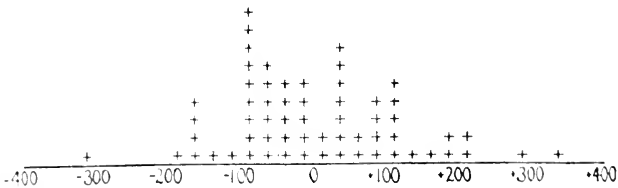
Algebraic Computation	20.42
Interpolation	91.93
Geometry	11.96
Superposition	12.03
Mixed Relations	20.59
Trabue Scales L and J	16.42

To simplify calculation the nearest integer is recommended for use, however, being sufficiently accurate.

[9] See page 66.

Individual	*Score*	*Individual*	*Score*
14	—179	44	—20
15	—72	45	—67
16	100	46	—95
17	121	47	—121
18	337	48	176
19	95	49	289
20	216	50	—171
21	12	51	—93
22	—3	52	182
23	218	53	—57
24	103	54	0
25	83	55	—163
26	38	56	—70
27	—34	57	31
28	—90	58	98
29	164	59	129
30	—29	60	—169
		61	—143

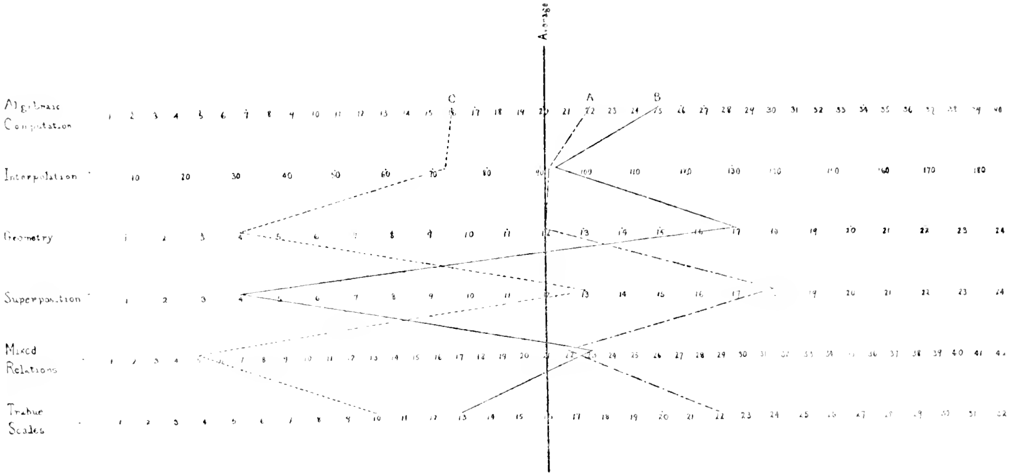
These scores are represented graphically in the accompanying figure.



As tentative standards we suggest (1) where a pupil's score is greater than 150, he has capacity to progress at a more rapid rate than the ordinary high school student; (2) where a pupil's score is less than —150, he shows incapacity to progress in mathematics at the rate of the ordinary high school student and, other things being equal, should be released from further training in the subject.

These standards are tentative, but they err on the safe side.[10]

[10] The reliability of the composite score for the sextet of tests has been estimated and is satisfactorily high. The results of two independent applications to the Horace Mann group gave a reliability coefficient of



Graph to Locate Special Mathematical Disability

Pupil A is above the average in all tests.
Pupil B is above the average in Algebraic Computation, Interpolation, Geometry, Mixed Relations; and below in Superposition and Trabue Scales.
Pupil C is below the average in all tests, save Superposition.

When any doubt is felt with regard to the ability of a pupil the tests should be re-applied in duplicate. Duplicates of the tests exist and can be supplied by the writer. More reliable standards will be established by application of the tests to larger numbers of children. Tests will be supplied at cost to those who will furnish results to the writer.

For the discovery of particular lines of mathematical weakness in the individual pupils use can be made of the graph to locate special disabilities, which is given on page 117. It provides a record of the relations of a pupil's abilities in mathematics to the abilities of others.

Each test is represented in the graph by a horizontal line. The scales are so drawn that the average marks for the six tests lie on a straight line. In the case of the Interpolation test all the scores are not directly indicated, but they can be roughly placed, when the individual curve is drawn. The interpretation of curves is simple. For example, in the graph, pupil A is seen to be above the average in all the tests, excelling especially in the test of intuitive grasp of spatial relations. Pupil C, on the other hand, is below the average in all save the Superposition test, failing conspicuously in ability to grasp spatial and abstract relations. Pupil B is above the average to a slight extent save in the Superposition test and the Trabue Language Scales, although he is not especially weak in the latter.

The graph to locate special disabilities can thus be profitably used as a check upon the opinions arrived at by the mathematics teacher as to the pupil's lines of strength and weakness.

.92 ± .01. Here *r* is seventy times as large as its Probable Error. Its reliability, therefore, is very high.

In the case of the Horace Mann [illegible] the equation [illegible] X = 12.44, [illegible] X = 12.36 [illegible] .02 [illegible] 1.01 [illegible] Yule, G. U. [illegible] London, 1916, 177.